Due at the beginning of the third tutorial, monday October 9th.

Problems titled [**SELF**] are for your own practice and will not be marked.

1. (a) [**10 pts.**] Prove using mathematical induction that the following is true for all positive integers n:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

   (b) [**4 pts.**] From the above result, infer whether the infinite series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$$

   converges or not. If so, what is the sum of this infinte series?

2. Please submit hand-drawn plots. You might want to first draw rough versions as you figure out what the graphs should look like, before copying the final version on to your submission.

   (a) [**5 pts.**] Plot the following three functions on the same graph:

$$f(x) = x^2 \quad \text{and} \quad g(x) = -x^2 \quad \text{and} \quad h(x) = (x-2)^2$$

   Your plot should extend roughly from -5 to $+5$.

   (b) [**5 pts.**] Plot the following two functions on the same graph:

$$f(x) = x^3 \quad \text{and} \quad g(x) = x^3 + 3$$

   Your plot should extend roughly from -2 to $+2$.

   (c) [**5 pts.**] Plot the following two functions on the same graph:

$$f(x) = x^2 \quad \text{and} \quad g(x) = x^2 + \frac{1}{x}$$

   Your plot should extend roughly from -5 to $+5$.

   (d) [**5 pts.**] Plot the following two functions on the same graph:

$$f(x) = x \quad \text{and} \quad g(x) = x - e^{-x}$$

   Your plot should extend roughly from -5 to $+5$.

3. Assume $|x| < 1$.

   (a) **[5 pts.]** Consider the infinite series

   $$P = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

   Subtract xP from P. Hence calculate P.

   (b) **[5 pts.]** Consider the infinite series

   $$Q = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{n=1}^{\infty} nx^n$$

   Subtract xQ from Q. Hence calculate Q.

   (c) **[3 pts.]** Is P a geometric series? If so, put it in the form ar^{n-1} by determining a and r. If not, explain why not.

   (d) **[3 pts.]** Is Q a geometric series? If so, put it in the form ar^{n-1} by determining a and r. If not, explain why not.

   (e) **[SELF]** Show that the ratio

   $$\frac{P}{Q} = \frac{1 + x + x^2 + x^3 + \cdots}{x + 2x^2 + 3x^3 + 4x^4 + \cdots}$$

   is equal to $\dfrac{x}{1-x}$.

4. (a) **[SELF]** Prove using mathematical induction that

   $$a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{r^n - 1}{r - 1}$$

   for any positive integer n. (We proved this in class using a completely different method.)

   (b) **[SELF]** Using the above result, explain why the infinite series

   $$a + ar + ar^2 + \cdots$$

   diverges whenever $|r| > 1$, but convereges whenever $|r| < 1$.

5. (a) **[SELF]** Prove using mathematical induction that

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) \;=\; n^2$$

for any positive integer n.

(b) **[SELF]** Prove the same result using the formula we derived in class for the sum of an arithmetic series.

(c) **[SELF]** Use the result above to argue whether the infinite series

$$1 + 3 + 5 + 7 + \cdots$$

converges or diverges.