1. (a) Find a formula for the n-th term of the arithmetic sequence

$$107,\ 98,\ 89,\ 80,\ ....$$

[6 marks]

(b) Sketch a plot of the finite sequence $\{a_n\}_{n=1}^{10}$ defined by

$$a_n \;=\; (-1)^n\, n$$

Does the infinite sequence $\{a_n\}_{n=1}^{\infty}$ converge?

[6 marks]

(c) Evaluate the sum of the first 100 odd positive integers:

$$1 + 3 + 5 + \cdots + 199$$

[6 marks]

(d) Evaluate the infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{4^n}$$

[7 marks]

2. (a) Sketch plots of the following three functions on the same graph:

$$f(x) = x^2 \quad ; \qquad g(x) = -x^2 \quad ; \qquad h(x) = (x-2)^2 \quad .$$

Your plot should extend roughly from -4 to $+4$, and each curve should be marked clearly as $f(x)$, $g(x)$, or $h(x)$.

[7 marks]

(b) Differentiate the following functions with respect to x.

(i) $x + 2\sqrt{x}$ (ii) $x^2 e^x$ (iii) e^{x^2-x}

[18 marks]

3. (a) Find the critical points of the function

$$f(x) \;=\; 2x + \frac{2}{x}$$

and determine whether each is a local maximum or a local minimum.

[15 marks]

(b) Calculate the definite integral

$$\int_{-2}^{2} x^3 dx \,.$$

By sketching a plot of the function being integrated and identifying the area represented by the integral, explain your result geometrically.

[10 marks]

4. (a) Obtain an approximation for the quantity $\sqrt{99}$, using the binomial expansion.

[9 marks]

(b) Given the matrices

$$M = \begin{pmatrix} p & q \end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \, ,$$

calculate MN and NM.

[8 marks]

(c) A particle moves along a straight line. If its position as a function of time is given by

$$x(t) \;=\; 2 - 4t + 3t^2$$

find the velocity and acceleration as functions of time.

[8 marks]

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SAMPLE SOLUTIONS

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1. (a) Find a formula for the n-th term of the arithmetic sequence

$$107, 98, 89, 80, ...$$

[6 marks]

[Sample Solution:]
Standard form for arithmetic sequences: $a_n = a + (n-1)d$. Here

$$a_1 = 107, \quad a_2 = 98$$

which gives $a = 107$ and $d = -9$. Thus the n-th term is

$$a_n = 107 + (n-1)(-9) = 116 - 9n$$

Note that $d = -9$, not $+9$!! Make sure you figure out why.
It would be wise to check immediately that this is the correct formula, by reproducing the next given terms of the sequence.

$$a_3 = 116 - 9(3) = 116 - 27 = 89$$
$$a_4 = 116 - 9(4) = 116 - 36 = 80$$

The predictions match the given values for the third and fourth members of the sequence. This gives us confidence that the formula is correct.

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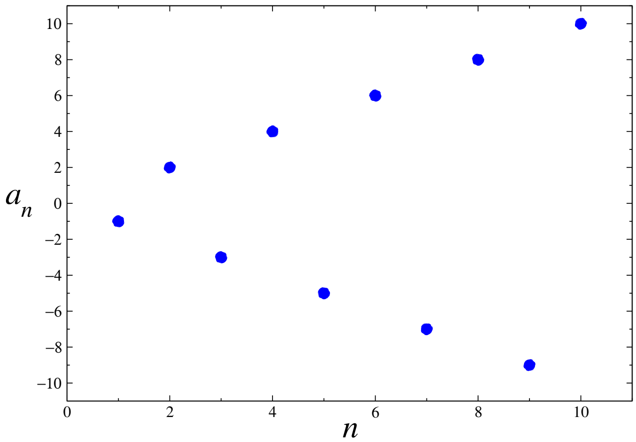
(b) Sketch a plot of the finite sequence $\{a_n\}_{n=1}^{10}$ defined by

$$a_n = (-1)^n n$$

Does the infinite sequence $\{a_n\}_{n=1}^{\infty}$ converge?

[6 marks]

[Sample Solution:]



You could think of joining the points, but does that really make sense? The sequence is only defined for integer values of n. So, strictly speaking, the plot should be a sequence of disjoint dots.

Also, note that you are asked to plot the *finite* sequence starting at $n = 1$ and ending at $n = 10$. So there should be no more points after $n = 10$.

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(c) Evaluate the sum of the first 100 odd positive integers:

$$1 + 3 + 5 + \cdots + 199$$

[6 marks]

[Sample Solution:]

$$\text{sum} \;=\; \frac{1}{2}\times(\#\text{ terms})\times(\text{first term}+\text{last term}) \;=\; \frac{1}{2}\times 100\times(1+199) \;=\; 10000$$

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(d) Evaluate the infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{4^n}$$

[7 marks]

[Sample Solution:]
Using the result

$$a + ar + ar^2 + \cdots \ = \ \frac{a}{1-r}$$

one obtains

$$\sum_{n=1}^{\infty} \frac{1}{4^n} \ = \ \frac{1}{4} \ + \ \frac{1}{4^2} \ + \ \frac{1}{4^3} \ + \ \cdots \ = \ \frac{\frac{1}{4}}{1 - \frac{1}{4}} \ = \ \frac{1}{3}$$

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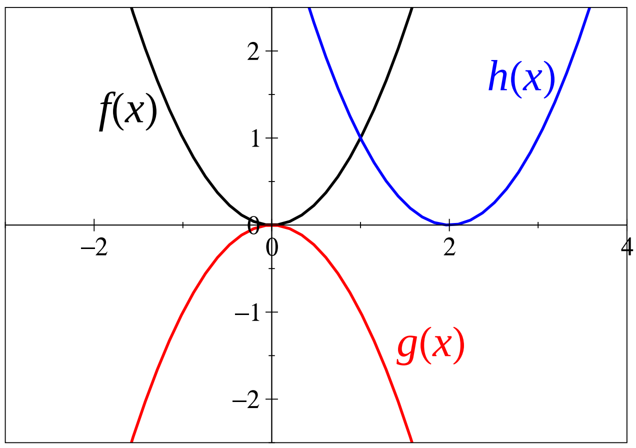
2. (a) Sketch plots of the following three functions on the same graph:

$$f(x) = x^2 \quad ; \qquad g(x) = -x^2 \quad ; \qquad h(x) = (x-2)^2 \quad .$$

Your plot should extend roughly from -4 to $+4$, and each curve should be marked clearly as $f(x)$, $g(x)$, or $h(x)$.

[7 marks]

[Sample Solution:]



This question checks for understanding of what happens to plots when functions are inverted or shifted.

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(b) Differentiate the following functions with respect to x.

(i) $x + 2\sqrt{x}$ (ii) $x^2 e^x$ (iii) e^{x^2-x}

[18 marks]

[Sample Solution:]

$$\begin{aligned}
\frac{d}{dx}\left(x + 2\sqrt{x}\right) &= 1 + \frac{1}{\sqrt{x}} \\
\frac{d}{dx}x^2 e^x &= x^2 \frac{d}{dx}e^x + \left(\frac{d}{dx}x^2\right)e^x = (x^2 + 2x)e^x \\
\frac{d}{dx}e^{x^2-x} &= e^{x^2-x} \times \frac{d}{dx}(x^2 - x) = e^{x^2-x}(2x-1)
\end{aligned}$$

The most important thing to have learned in this module was differentiation. This is also the main skill tested in the PD155 exam.

If you are not able to derivatives like this correctly and swiftly, you probably should learn how to differentiate, and/or get help, and spend a couple of days practicing.

If you are not absolutely sure about using the product rule and the chain rule, the rest of the exam will probably be too difficult for you.

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3. (a) Find the critical points of the function

$$f(x) \;=\; 2x + \frac{2}{x}$$

and determine whether each is a local maximum or a local minimum.

[15 marks]

[**Sample Solution:**]

$$f'(x) \;=\; 2 - \frac{2}{x^2} \qquad f''(x) \;=\; +\frac{4}{x^3}$$

Setting $f'(x) = 0$ gives the two critical points $x = \pm 1$.
For $x = -1$:

$$f''(-1) \;=\; -4 < 0 \qquad \Longrightarrow \qquad \text{minimum}$$

For $x = +1$:

$$f''(+1) \;=\; +4 > 0 \qquad \Longrightarrow \qquad \text{maximum}$$

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(b) Calculate the definite integral

$$\int_{-2}^{2} x^3 dx\,.$$

By sketching a plot of the function being integrated and identifying the area represented by the integral, explain your result geometrically.

[10 marks]

[Sample Solution:]

$$\int_{-2}^{2} x^3 dx \;=\; \left[\frac{x^4}{4}\right]_{-2}^{+2} \;=\; \frac{(+2)^4}{4} - \frac{(-2)^4}{4} \;=\; 0$$

The plot and/or geometric argument should show that the relevant area has equal negative and positive parts, and hence add up to zero.

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4. (a) Obtain an approximation for the quantity $\sqrt{99}$, using the binomial expansion.

[9 marks]

[**Sample Solution:**]

$$\begin{aligned}
\sqrt{99} &= (100-1)^{1/2} = 100^{1/2}\left(1-\frac{1}{100}\right)^{1/2} \\
&= 10\left(1-\frac{1}{2}\frac{1}{100}+\text{further terms}\right) \approx 10\left(1-\frac{1}{2}\frac{1}{100}\right) \\
&= 10(1-0.005) = 10\times 0.995 = 9.95
\end{aligned}$$

Two examples of such approximation were worked out in class.

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(b) Given the matrices

$$M = \begin{pmatrix} p & q \end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \ ,$$

calculate MN and NM.

[8 marks]

[**Sample Solution:**]

$$\begin{aligned} MN &= \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = p\alpha + q\beta \\ NM &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} p & q \end{pmatrix} = \begin{pmatrix} \alpha p & \alpha q \\ \beta p & \beta q \end{pmatrix} \end{aligned}$$

Please learn matrix multiplication. This means, you should be able to answer:
When can a matrix with R rows and S columns be multiplied with a matrix with V rows and W columns? If they can be mutiplied, how many rows and columns does the resulting matrix have?

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(c) A particle moves along a straight line. If its position as a function of time is given by

$$x(t) \;=\; 2 - 4t + 3t^2$$

find the velocity and acceleration as functions of time.

[8 marks]

[Sample Solution:]

The velocity is the derivative of the displacement (position) with respect to time.

The acceleration is the derivative of the velocity with respect to time.

$$\begin{aligned} v(t) &= \frac{d}{dt}x(t) \;=\; \frac{d}{dt}(2 - 4t + 3t^2) \;=\; -4 + 6t \\ a(t) &= \frac{d}{dt}v(t) \;=\; \frac{d}{dt}(-4 + 6t) \;=\; 6 \end{aligned}$$

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