OLLSCOIL NA hÉIREANN MÁ NUAD
THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

MATHEMATICAL PHYSICS

AUTUMN REPEAT EXAMINATION
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Condensed Matter Theory
Interactions, Magnetism and Superconductivity
MP473

Prof. S. J. Hands, Dr. J. K. Slingerland and Dr. J.-I. Skullerud

Time allowed: $1\frac{1}{2}$ hours
Answer ALL questions

1. A 1-dimensional magnet consists of N spin-$\frac{1}{2}$ particles, interacting with each other and with a magnetic field through the following Hamiltonian,

$$H = -J \sum_{i=0}^{N-2} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y\right) - h \sum_{i=0}^{N-1} \sigma_l^z,$$

where J and h are real constants.
We define fermionic creation and annihilation operators in terms of the spin operators by the following Jordan-Wigner formula,

$$c_l = \frac{1}{2}\left(\prod_{j<l} \sigma_j^z\right)(\sigma_l^x + i\sigma_l^y) \qquad c_l^\dagger = \frac{1}{2}\left(\prod_{j<l} \sigma_j^z\right)(\sigma_l^x - i\sigma_l^y)$$

It is given that these satisfy the canonical anticommutation relations for fermionic creation and annihilation operators.

   (a) Derive that the Hamiltonian can be rewritten as,

$$H = -2J \sum_{l=0}^{N-2} \left(c_l^\dagger c_{l+1} - c_l c_{l+1}^\dagger\right) + h(2N_F - N)$$

   where N_F is the total number of fermions in the system **[20 marks]**

   We have $c_l + c_l^\dagger = \left(\prod_{j<l} \sigma_j^z\right) \sigma_l^x$ and hence $\sigma_l^x = \left(\prod_{j<l} \sigma_j^z\right)(c_l + c_l^\dagger)$. Similarly, we find that $\sigma_l^y = i\left(\prod_{j<l} \sigma_j^z\right)(c_l^\dagger - c_l)$. We also have $c_l^\dagger c_l = \frac{1}{4}(\sigma_l^x - i\sigma_l^y)(\sigma_l^x + i\sigma_l^y)) = \frac{1}{2}(1-\sigma_l^z)$, so $\sigma_l^z = 1 - 2c_l^\dagger c_l = 1 - 2\hat{n}_l$. This gives

$$\sigma_l^x \sigma_{l+1}^x = (c_l + c_l^\dagger)\sigma_l^z(c_{l+1} + c_{l+1}^\dagger) = -(c_l + c_l^\dagger)(2c_l^\dagger c_l - 1)(c_{l+1} + c_{l+1}^\dagger)$$

   and

$$\sigma_l^y \sigma_{l+1}^y = -(c_l^\dagger - c_l)\sigma_l^z(c_{l+1}^\dagger - c_{l+1}) = (c_l^\dagger - c_l)(2c_l^\dagger c_l - 1)(c_{l+1}^\dagger - c_{l+1})$$

   Noting that

$$\begin{aligned} c_l^\dagger(2c_l^\dagger c_l - 1) &= -c_l^\dagger \\ c_l(2c_l^\dagger c_l - 1) = 2(\{c_l, c_l^\dagger\} - c_l^\dagger c_l)c_l - c_l &= c_l \end{aligned}$$

   we find that

$$\begin{aligned} \sigma_l^x \sigma_{l+1}^x &= (c_l^\dagger - c_l)(c_{l+1} + c_{l+1}^\dagger) = -c_l c_{l+1} - c_l c_{l+1}^\dagger + c_l^\dagger c_{l+1} + c_l^\dagger c_{l+1}^\dagger \\ \sigma_l^y \sigma_{l+1}^y &= -(c_l^\dagger + c_l)(c_{l+1}^\dagger - c_{l+1}) = c_l c_{l+1} - c_l c_{l+1}^\dagger + c_l^\dagger c_{l+1} - c_l^\dagger c_{l+1}^\dagger \end{aligned}$$

   and further using that $\sum_{i=0}^{N-1} \hat{n}_i = N_F$, we see that the Hamiltonian has the required form.

(b) We now introduce the Fourier transformed raising and lowering operators

$$d_k = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} c_l \, e^{2\pi i\, lk/N} \qquad d_k^\dagger = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} c_l^\dagger e^{-2\pi i\, lk/N}$$

Show that these operators satisfy the canonical anticommutation relations for fermionic raising and lowering operators. **[15 marks]**

We have

$$\{d_k, d_k^\dagger\} = \frac{1}{N} \sum_{l,l'=0}^{N-1} \{c_l\,, c_{l'}^\dagger\} e^{2\pi i(lk-l'k')/N} = \frac{1}{N} \sum_{l=0}^{N-1} e^{2\pi il(k-k')/N} = \delta_{k,k'}$$

We used firstly $\{c_l\,, c_{l'}^\dagger\} = \delta_{l,l'}$ (it was given that the c_i satisfy the canonical anticommutation relations) and secondly $\sum_{l=0}^{N-1} e^{2\pi il(k-k')/N} = N\delta_{k,k'}$. We can also see directly from $\{c_l\,, c_{l'}\} = 0$ that $\{d_k, d_{k'}\} = 0$.

(c) We now change the Hamiltonian of the fermionic system slightly by including a coupling between the beginning and end of the chain (so it is effectively a ring). To write this in a convenient way we define $c_N := c_0$. The new Hamiltonian is $\tilde{H} = -2J \sum_{l=0}^{N-1} \left(c_l^\dagger c_{l+1} - c_l \, c_{l+1}^\dagger \right) + h(2N_F - N)$. Show that $\tilde{H} = -4J \sum_{k=0}^{N-1} \cos(2\pi k/N) d_k^\dagger d_k + h(2N_F - N)$.
Find the energy of the ground state and first excited state of the system when $h > 2J > 0$. **[15 marks]**

The term $h(2N_F - N)$ does not change under Fourier transformation. For the rest of the Hamiltonian, we have

$$\begin{aligned}
\tilde{H} &= -2J/N \sum_{l,k,k'} \left(d_k^\dagger d_{k'} e^{2\pi ilk/N} e^{-2\pi i(l+1)k'/N} - d_k d_{k'}^\dagger e^{-2\pi ilk/N} e^{+2\pi i(l+1)k'/N} \right) \\
&= -2J/N \sum_{k,k'} \left(e^{-2\pi ik'} d_k^\dagger d_{k'} \sum_l e^{2\pi il(k-k')/N} - e^{+2\pi ik'} d_k d_{k'}^\dagger \sum_l e^{2\pi il(k'-k)/N} \right) \\
&= -2J \sum_k \left(e^{-2\pi ik/N} d_k^\dagger d_k - e^{2\pi ik/N} d_k d_k^\dagger \right) \\
&= -4J \sum_{k=0}^{N-1} \cos(2\pi k/N) d_k^\dagger d_k + 2J \sum_k e^{2\pi ik/N} = -4J \sum_{k=0}^{N-1} \cos(2\pi k/N) d_k^\dagger d_k
\end{aligned}$$

We now note that if $h > 2J > 0$, every fermion that is present gives a positive contribution to the energy. Concretely, a fermion with wave number k contributes energy $2h - 4J\cos(2\pi k/N) > 2h - 4J > 0$. Therefore, in the ground state, there are no fermions ($N_F = \hat{n}_k = 0$ for all k) and we find $E_0 = -hN$. In the first excited state there will be one fermion. The minimal energy cost is achieved when this fermion has $k = 0$. We then have $E_1 = -hN + 2(h - 2J)$. The system becomes gapless at $h = 2J$

2. A system of spinless (or spin polarized) fermions of mass m in one space dimension has the following Hamiltonian

$$H = \sum_k \frac{\hbar^2 k^2}{2m} c_k^\dagger c_k + \sum_{k,k',q,n} \lambda_n q^{2n} c_{k+q}^\dagger c_{k'-q}^\dagger c_{k'} c_k$$

The fermions are confined to a line segement of length L with periodic boundary conditions. Hence the wave numbers k, k' and q which appear are all integer multiples of $\frac{2\pi}{L}$. The sum over n runs over all nonnegative integers and the λ_n are coupling constants.

(a) In this part, we set the coupling constants λ_n equal to zero for all n. Let $\{k_1, \ldots, k_N\}$ be a set of N wave numbers.
Show that the state $\prod_{i=1}^N c_{k_i}^\dagger |0\rangle$ is an eigenstate of H and find its energy. **[15 marks]**

All we need is to find the action of $\hat{n}_k = c_k^\dagger c_k$ on the given state, for all k. If $k \notin \{k_1, \ldots, k_N\}$ then c_k anticommutes with $c_{k_i}^\dagger$ for all i and we find $c_k^\dagger c_k \prod_{i=1}^N c_{k_i}^\dagger |0\rangle = (-1)^N c_k^\dagger \prod_{i=1}^N c_{k_i}^\dagger c_k |0\rangle = 0$, since $c_k|0\rangle = 0$ for all k. If $k = k_j$ for some j, then we have

$$c_{k_j}^\dagger c_{k_j} \prod_{i=1}^N c_{k_i}^\dagger |0\rangle = \left(\prod_{i=1}^{j-1} c_{k_i}^\dagger\right) c_{k_j}^\dagger c_{k_j} c_{k_j}^\dagger \left(\prod_{i=j+1}^{N} c_{k_i}^\dagger\right) |0\rangle = \prod_{i=1}^N c_{k_i}^\dagger |0\rangle$$

The last equality follows from $c_{k_j}^\dagger c_{k_j} c_{k_j}^\dagger = (\{c_{k_j}^\dagger, c_{k_j}\} - c_{k_j} c_{k_j}^\dagger) c_{k_j}^\dagger = c_{k_j}^\dagger$, which itself follows from $(c_{k_j}^\dagger)^2 = 0$ and $\{c_{k_j}^\dagger, c\ \} = 1$. Thus all these states are eigenstates of the $c_k^\dagger c_k$ which means they are also eigenstates of H and we read off that the eigenvalue of the given state is $\sum_i \frac{\hbar^2 k_i^2}{2m}$

(b) We now consider the case with $\lambda_n \neq 0$ for all $n \geq 0$. We treat the new nonzero terms in the Hamiltonian as a perturbation. Show that the correction to the energy of the states considered in part **(a)** in first order perturbation theory is given by

$$\Delta E = -\sum_{k,k' \in \{k_1, \ldots, k_N\}} \sum_{n=1}^{\infty} \lambda_n (k' - k)^{2n}$$ **[20 marks]**

The correction to the energy is simply the expectation value of the interaction terms in H in the unperturbed eigenstates $\prod_{i=1}^N c_{k_i}^\dagger |0\rangle$. The expectation value of a term $q^{2n} c_{k+q}^\dagger c_{k'-q}^\dagger c_{k'} c_k$ can be nonzero only if all of $k, k', k+q$ and $k'-q$ are elements of $\{k_1, \ldots, k_N\}$ since otherwise either c_k or $c_{k'}$ commute to the right through $\prod_{i=1}^N c_{k_i}^\dagger$ to act on $|0\rangle$ (which gives 0) or $c_{k+q}^\dagger$

or $c^\dagger_{k'-q}$ similarly commute to the left to act on $\langle 0|$ (again giving 0). In fact, we must have $\{k+q, k'-q\} = \{k, k'\}$ to get a nonzero expectation value, i.e. the interaction must annihilate and create particles at the same momenta, or the initial and final state will have different occupation numbers and they will have overlap zero. This can also be shown by direct use of the anticommutation relations, very similarly to the computation in part **(a)**. As a result we must have either $k = k+q$ and $k' = k'-q$ and hence $q = 0$ (this possibility gives zero whenever $n > 0$ due to the factor q^{2n}), or $k' = k+q$ and $k = k'-q$ and hence $q = k'-k$. The sum over q is then reduced to only two terms. The sums over k and k' are reduced to sums over the set $\{k_1, \ldots, k_N\}$. We have

$$\begin{aligned}
\Delta E &= \sum_{k,k'} \left(\lambda_0 \langle \psi | c^\dagger_k c^\dagger_{k'} c_{k'} c_k |\psi\rangle + \sum_{n=0}^{\infty} \lambda_n (k'-k)^{2n} \langle \psi | c^\dagger_{k'} c^\dagger_k c_{k'} c_k |\psi\rangle \right) \\
&= \sum_{k \neq k'} \left(\lambda_0 \langle \psi | \hat{n}_k \hat{n}_{k'} |\psi\rangle - \sum_{n=0}^{\infty} \lambda_n (k'-k)^{2n} \langle \psi | \hat{n}_k \hat{n}_{k'} |\psi\rangle \right) \\
&= -\sum_{k,k' \in \{k_1,\ldots,k_N\}} \sum_{n=1}^{\infty} \lambda_n (k'-k)^{2n}
\end{aligned}$$

as claimed. Notice that in the last step, we used that the occupation numbers n_k, $n_{k'}$ equal 1 in the state ψ for $k, k' \in \{k_1, \ldots, k_N\}, k \neq k$. Also the λ_0 terms cancel and after that we can drop the requirement that $k \neq k'$ since all remaining terms equal zero when $k = k'$.

(c) We now consider a system which has $\lambda_1 = V/L$ for some constant $V > 0$ and $\lambda_n = 0$ for $n > 1$.
Calculate the expectation value of the energy per particle in the ground state of the non-interacting system, at large N. Express the result in terms of the mass m, the constant V and the particle density $n = \frac{N}{L}$

[15 marks]

The ground states of the non-interacting system are superpositions of the states $\prod_{l=-N/2+1}^{N/2} c^\dagger_{2\pi l\hbar/L}$ and $\prod_{l=-N/2}^{N/2-1} c^\dagger_{2\pi l\hbar/L}$, when N is even, while if N is odd there is a unique ground state, $\prod_{l=-(N-1)/2}^{(N-1)/2} c^\dagger_{2\pi l\hbar/L}$. Either way, the kinetic energy per particle is, to good approximation, given by

$$\frac{E_{kin}}{N} \approx \frac{L}{2\pi N} \int_{-\pi N/L}^{\pi N/L} \frac{\hbar^2 k^2}{2m} dk = \frac{L\hbar^2}{4\pi m N} \frac{2}{3} (\frac{\pi N}{L})^3 = \frac{\hbar^2 \pi^2}{6m} n^2.$$

The potential energy is given by the formula in part **(b)**, which reduces to

$$\begin{aligned}
\frac{\Delta E}{N} &= -\frac{\lambda_1}{N}\sum_{k,k'=-(N/2)(2\pi/L)}^{(N/2)(2\pi/L)} (k'-k)^2 \approx -\frac{V}{NL}\frac{L^2}{4\pi^2}\int_{-\frac{N\pi}{L}}^{\frac{N\pi}{L}} dk \int_{-\frac{N\pi}{L}}^{\frac{N\pi}{L}} dk'(k'-k)^2 \\
&= -\frac{VL}{12\pi^2 N}\int_{-\frac{N\pi}{L}}^{\frac{N\pi}{L}} dk \left\{ (\frac{N\pi}{L}-k)^3 - (-\frac{N\pi}{L}-k)^3 \right\} \\
&= -\frac{VL}{48\pi^2 N}\left[(\frac{N\pi}{L}-k)^4 - (-\frac{N\pi}{L}-k)^4 \right]_{k=-\frac{N\pi}{L}}^{k=\frac{N\pi}{L}} \\
&= \frac{VL}{24\pi^2 N}\left(\frac{2N\pi}{L}\right)^4 = \frac{2\pi^2}{3}V\left(\frac{N}{L}\right)^3 = \frac{2\pi^2 V}{3}n^3
\end{aligned}$$