OLLSCOIL NA hÉIREANN MÁ NUAD

THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

MATHEMATICAL PHYSICS

AUTUMN REPEAT EXAMINATION

2016-2017

Condensed Matter Theory

Interactions, Magnetism and Superconductivity

MP473

Prof. S. J. Hands, Dr. J. K. Slingerland and Dr. J.-I. Skullerud

Time allowed: $1\frac{1}{2}$ hours

Answer ALL questions

1. A 1-dimensional magnet consists of N spin-$\frac{1}{2}$ particles, interacting with each other and with a magnetic field through the following Hamiltonian,

$$H = -J \sum_{i=0}^{N-2} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y\right) - h \sum_{i=0}^{N-1} \sigma_l^z,$$

where J and h are real constants.
We define fermionic creation and annihilation operators in terms of the spin operators by the following Jordan-Wigner formula,

$$c_l = \frac{1}{2}\left(\prod_{j<l} \sigma_j^z\right)(\sigma_l^x + i\sigma_l^y) \qquad c_l^\dagger = \frac{1}{2}\left(\prod_{j<l} \sigma_j^z\right)(\sigma_l^x - i\sigma_l^y)$$

It is given that these satisfy the canonical anticommutation relations for fermionic creation and annihilation operators.

(a) Derive that the Hamiltonian can be rewritten as,

$$H = -2J \sum_{l=0}^{N-2} \left(c_l^\dagger c_{l+1} - c_l c_{l+1}^\dagger\right) + h(2N_F - N)$$

where N_F is the total number of fermions in the system **[20 marks]**

(b) We introduce the Fourier transformed raising and lowering operators

$$d_k = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} c_l e^{2\pi i lk/N} \qquad d_k^\dagger = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} c_l^\dagger e^{-2\pi i lk/N}$$

Show that these operators satisfy the canonical anticommutation relations for fermionic raising and lowering operators. **[15 marks]**

(c) We now change the Hamiltonian of the fermionic system slightly by including a coupling between the beginning and end of the chain (so it is effectively a ring). To write this in a convenient way we define $c_N := c_0$. The new Hamiltonian is $\tilde{H} = -2J\sum_{l=0}^{N-1}\left(c_l^\dagger c_{l+1} - c_l c_{l+1}^\dagger\right) + h(2N_F - N)$.
Show that $\tilde{H} = -4J\sum_{k=0}^{N-1}\cos(2\pi k/N) d_k^\dagger d_k + h(2N_F - N)$.
Find the energy of the ground state and first excited state of the system when $h > 2J > 0$. **[15 marks]**

2. A system of spinless (or spin polarized) fermions of mass m in one space dimension has the following Hamiltonian

$$H = \sum_k \frac{\hbar^2 k^2}{2m} c_k^\dagger c_k + \sum_{k,k',q,n} \lambda_n q^{2n} c_{k+q}^\dagger c_{k'-q}^\dagger c_{k'} c_k$$

The fermions are confined to a line segment of length L with periodic boundary conditions. Hence the wave numbers k, k' and q which appear are all integer multiples of $\frac{2\pi}{L}$. The sum over n runs over all nonnegative integers and the λ_n are coupling constants.

(a) In this part, we set the coupling constants λ_n equal to zero for all n.
Let $\{k_1, \ldots, k_N\}$ be a set of N wave numbers.
Show that the state $\prod_{i=1}^N c_{k_i}^\dagger |0\rangle$ is an eigenstate of H and find its energy.
[15 marks]

(b) We now consider the case where the λ_n may be nonzero for all $n \geq 0$. We treat the new nonzero terms in the Hamiltonian as a perturbation. Show that the correction to the energy of the states considered in part **(a)** in first order perturbation theory is given by

$$\Delta E = -\sum_{k,k' \in \{k_1,\ldots,k_N\}} \sum_{n=1}^{\infty} \lambda_n (k' - k)^{2n}$$ **[20 marks]**

(c) We now consider a system which has $\lambda_1 = V/L$ for some constant $V > 0$ and $\lambda_n = 0$ for $n > 1$.
Calculate the expectation value of the energy per particle in the ground state of the non-interacting system, at large N. Express the result in terms of the mass m, the constant V and the particle density $n = \frac{N}{L}$
[15 marks]