MP472 EXAM 2019-2020

(1a) (i) Consider the initial state $|10\rangle$ and show how the circuit $CNOT_{12}H_1$ transform this state.
(ii) Determine the matrix representation of the circuit in the standard computational basis.

Solution:
(i) $CNOT_{12}H_1\,|10\rangle = CNOT_{12}(|00\rangle - |10\rangle)/\sqrt{2} = (|00\rangle - |11\rangle)/\sqrt{2}$
(ii)

$$\hat{\rho} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \tag{1}$$

(1b) Calculate how the bit-flip X, and phase-flip Z, and the Hadamard gate H, transform the Bloch vector of the states
(i) $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$,
(ii) $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$.

Solution:
(i) The state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ in the Bloch representation is given by the decomposition of the relevant density operator as

$$\rho = \frac{1}{2}.I + \frac{1}{2}.X \tag{2}$$

The action of the operators onto the state is then given as follows

$$X\rho X = \frac{1}{2}.I + \frac{1}{2}.X \tag{3}$$

$$Z\rho Z = \frac{1}{2}.I - \frac{1}{2}.X \tag{4}$$

$$H\rho H = \frac{1}{2}.I + \frac{1}{4}.(X+Z)X(X+Z) = \frac{1}{2}.I + \frac{1}{4}.X - \frac{1}{4}.X + \frac{1}{2}.Z = \frac{1}{2}.I + \frac{1}{2}.Z \tag{5}$$

(ii) Similarly

$$\rho = \frac{1}{2}.I - \frac{1}{2}.Y \tag{6}$$

The action of the operators onto the state is then given as follows

$$X\rho X = \frac{1}{2}.I + \frac{1}{2}.Y \tag{7}$$

$$Z\rho Z = \frac{1}{2}.I + \frac{1}{2}.Y \tag{8}$$

$$H\rho H = \frac{1}{2}.I - \frac{1}{4}.(X+Z)Y(X+Z) = \frac{1}{2}.I + \frac{1}{2}.Y \tag{9}$$

(1c) (i) Determine the matrix representation of the unitary transformation , $U_X(t) = \exp(itX)$, where X is the bit flip operator and i is the imaginary unit, and
(ii) calculate its effect on the state given by the density matrix

$$\hat{\rho} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

and comment on the result.

Solution: The unitary rotation is around the x axis in the Bloch representation and thus it leaves the state, which is an equal superposition $\frac{1}{2}(|0\rangle + |1\rangle)$, invariant. To see that we first decompose the unitary rotation as as

$$U_X(t) = \cos(t).I + i.\sin(t).X \tag{10}$$

and then

$$U_X(t)\rho U_X^\dagger(t) = (\cos(t).I + i.\sin(t).X)(\frac{1}{2}.I + \frac{1}{2}.X)(\cos(t).I - i.\sin(t).X) = \frac{1}{2}.I + \frac{1}{2}.X \tag{11}$$

as $X^2 = I$ and $cos(t)^2 + sin(t)^2 = 1$.

(2a) The model of classical computation is given by the Deterministic Turing Machine in which a computation proceeds through a single computation path. Explain in words how the Quantum Turing machine differs from the classical Turing machine and what consequences this difference has for power of quantum computation.

Solution:
Computation process with the quantum Turing machine proceeds through a multiplicity of computing paths each weighted by a probability amplitude. The probability of measuring correct result of computation is enhanced by constructive interference while incorrect results are suppressed. This quantum parallelism gives quantum computation exponential speedup for problems like factorization compared to any known classical computation and thus makes certain classically intractable problems tractable.

(2b) The nine-qubit phase-flip code allows correcting single qubit errors by encoding logical qubits as

$$\begin{aligned} |0\rangle_L &\equiv \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\ |1\rangle_L &\equiv \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle). \end{aligned}$$

(i) Describe what syndrome measurements need to be performed to detect a bit flip and phase flip errors on one of the qubits, and
(ii) explain what operation must be performed to correct the errors.

Solution:
(i) The following syndrom measurements detect the phase flip errors

$$X_1X_2X_3X_4X_5X_6, X_4X_5X_6X_7X_8X_9 \tag{12}$$

and the bit flip errors

$$Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9 \tag{13}$$

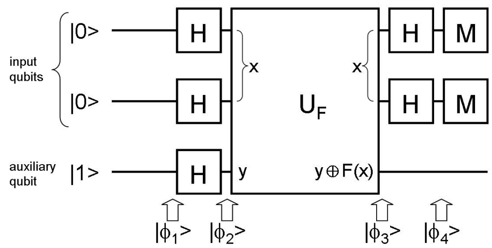
(ii) Phase flip errors can be corrected by the operators

$$Z_1Z_2Z_3, Z_4Z_5Z_6, Z_7Z_8Z_9 \tag{14}$$

and the bit flip errors by

$$X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9 \tag{15}$$

(2c) Deutsch-Jozsa algorithm is a quantum algorithm which decides whether a binary function (implemented as a black box or oracle) is *constant*, i.e. the function gives the same value for all inputs, or *balanced*, for a half of the inputs the function outputs 0. Figure below shows the quantum circuit for this algorithm.



Explain the algorithm how the algorithm works on the example of the two-qubit function which is given as f(00)=1, f(01)=0, f(10)=0, f(11)=1. That is, calculate the states at various steps of the circuit as shown above and explain the measurement result.

Solution:
The first set of Hadamard gates prepares the initial state $|\phi_1\rangle = |001\rangle$ in the superposition, with the auxiliary qubit factored out,

$$|\phi_2\rangle = \frac{1}{2^{3/2}}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)(|0\rangle - |1\rangle) \tag{16}$$

The oracle in a single call adds the functional values to the values of the auxiliary qubit mod(2), $y \oplus f(x)$. This changes the state as follows

$$|\phi_3\rangle = \frac{1}{2^{3/2}}(-|00\rangle + |01\rangle + |10\rangle - |11\rangle)(|0\rangle - |1\rangle) \tag{17}$$

To see the effect of the last two Hadamard transformation, the state can be rewritten as (the auxiliary qubit can be omitted)

$$|\phi_3\rangle = -\frac{1}{2^{3/2}}(|0\rangle(|0\rangle - |1\rangle) + |1\rangle(|0\rangle - |1\rangle)) \tag{18}$$

which yields after the Hadamard transformations the state $|01\rangle$ which will be detected with certainty by the independent measurement of both qubits.