Assignment 5: selected solutions

Problem 1: Compare the computational complexity of classical algorithm for distinguishing the constant and balanced n-bit functions, implemented as an oracle, with the complexity of the Deutsch-Jozsa quantum algorithm for the same problem.

Solution:
The n-bit function has 2^n possible inputs. Classically, more than half of the inputs needs to be tested to determine whether the function is constant of balanced, that is, the function has to be evaluated $2^{n-1}+1$ times. On a quantum computer, the function is for any n evaluated in 1 step as a unitary transformation applied to the state that represents a uniform superposition of all possible input values.

Problem 2: Consider that the oracle for the Deutsch-Jozsa quantum algorithm with two input qubits and one auxiliary qubit is given in the standard computational basis by the following matrix

$$\hat{U} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using the oracle above, calculate the standard matrix representation of the circuit for the Deutsch-Jozsa algorithm, then determine what outcome is to be obtained when measurement of the first two qubits is performed at the end of the circuit, and determine from this result what type of function, whether constant or balanced, the oracle implements.

Solution:

$$(H \otimes H \otimes \hat{1})(U)(H \otimes H \otimes H)|001\rangle = \frac{1}{2^{5/2}} \times$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

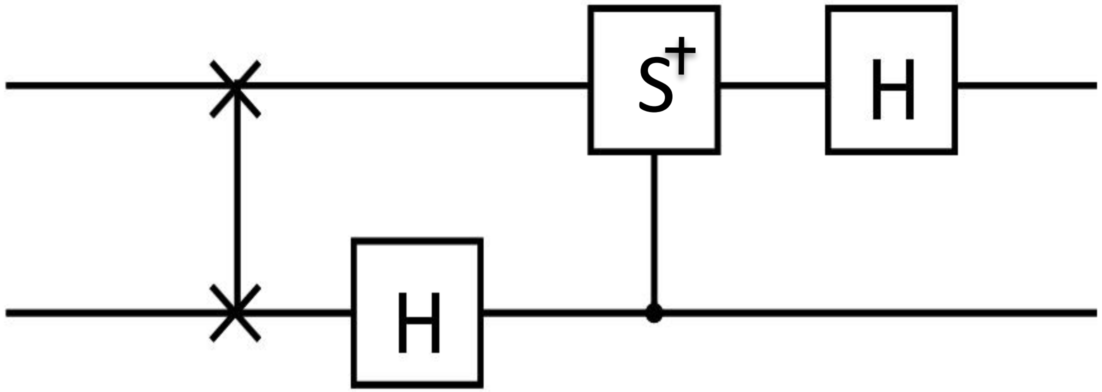
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|110\rangle - |111\rangle) = |11\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

Since the state of the first two qubits is

$$|11\rangle$$

their measurement will yield the result 1 for both qubits with the probability $p = 1$, and hence the oracle computed a balanced function.

Problem 3: Determine the matrix representation of the inverse Fourier transform given by the circuit:



Solution:

The inverse QFT circuit is the following sequence of operations

$$(H \otimes I)(\text{controlled-}S^{\dagger}_{21})(I \otimes H)(SWAP).$$

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This becomes in the matrix representation

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}.$$

The resulting matrix is indeed the inverse QFT:

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} = \hat{1}.$$

Problem 4: Consider the phase estimation algorithm. The state resulting from the first stage of the algorithm is given as

$$|\psi\rangle = \frac{1}{2}\Big(|0\rangle + e^{\pi i}|1\rangle\Big) \otimes \Big(|0\rangle + e^{\pi i/2}|1\rangle\Big).$$

Apply the inverse Fourier transform from the previous problem to the state $|\psi\rangle$ in the standard matrix representation, and determine the result of the phase estimation algorithm, that is, the binary representation of the phase φ.

Solution:

The state

$$|\psi\rangle = \frac{1}{2}\left(|0\rangle + e^{\pi i}|1\rangle\right) \otimes \left(|0\rangle + e^{\pi i/2}|1\rangle\right) = \frac{1}{2}\left(|00\rangle + e^{\pi i/2}|01\rangle + e^{\pi i}|10\rangle + e^{3\pi i/2}|11\rangle\right)$$

is in the standard matrix representation given as

$$|\psi\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ e^{\pi i/2} \\ e^{\pi i} \\ e^{3\pi i/2} \end{pmatrix}.$$

The action of the inverse QFT onto the given state produces

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}\begin{pmatrix} 1 \\ e^{\pi i/2} \\ e^{\pi i} \\ e^{3\pi i/2} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}\begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

which is the basis vector 01.

The result of the phase estimation algorithm is the phase given in the binary fraction notation as $\varphi = .01 = 2^{-1}.0 + 2^{-2}.1$ which is $\varphi = 0.25$ in the decadic notation.

Problem 5: Consider the Grover algorithm for searching unstructured two-qubit database. Choose the marked state in the database to be $|10\rangle$, and show by an explicit calculation that a single application of the Grover iteration returns the marked state, that is, the final state after the Grover iteration is $|\psi\rangle = |10\rangle$.

Solution:

$$
\begin{aligned}
|s\rangle &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\
U_\omega|s\rangle &= \frac{1}{2}(\hat{1} - 2|10\rangle\langle 10|)(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)
\end{aligned}
$$

$$U_s U_\omega|s\rangle = (2|s\rangle\langle s| - \hat{1})U_\omega|s\rangle$$

$$= \frac{1}{2}(-|00\rangle\langle 00| + |00\rangle\langle 01| + |00\rangle\langle 10| + |00\rangle\langle 11| + |01\rangle\langle 00| - |01\rangle\langle 01| + |01\rangle\langle 10| + |01\rangle\langle 11| + |10\rangle\langle 00| + |10\rangle\langle 01| - |10\rangle\langle 10| + |10\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 01| + |11\rangle\langle 10| - |11\rangle\langle 11|) \times \left[\frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)\right]$$

$$= |10\rangle.$$

Single application of the Grover iteration recovers the marked state exactly.

Quantum search algorithm (Grover)

Consider an unsorted database with $N = 2^n$ entries where n is the number of qubits. The problem is to determine the index of the database entry which satisfies some search criterion, that is, to identify the marked state $|\omega\rangle$.

We are provided with oracle access to a unitary operator, U_ω, which acts as follows:

$$\begin{aligned} U_\omega|\omega\rangle &= -|\omega\rangle \\ U_\omega|x\rangle &= |x\rangle, \text{ for all } x \neq \omega. \end{aligned}$$

The operator U_ω can be rewritten as

$$U_\omega = \hat{I} - 2|\omega\rangle\langle\omega|$$

$$\begin{aligned} (\hat{I} - 2|\omega\rangle\langle\omega|)\,|\omega\rangle &= |\omega\rangle - 2|\omega\rangle\langle\omega|\omega\rangle = -|\omega\rangle, \\ (\hat{I} - 2|\omega\rangle\langle\omega|)\,|x\rangle &= |x\rangle - |\omega\rangle\langle\omega|x\rangle = |x\rangle. \end{aligned}$$

Let $|s\rangle$ denote the uniform superposition over all states

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle$$

We introduce the Grover diffusion operator

$$U_s = 2|s\rangle\langle s| - \hat{I}.$$

The following computations show what happens in the first iteration:

$$\begin{aligned}
\langle s|\omega\rangle &= \frac{1}{\sqrt{N}} \\
\langle s|s\rangle &= N\frac{1}{\sqrt{N}} \cdot \frac{1}{\sqrt{N}} = 1 \\
U_\omega|s\rangle &= (\hat{I} - 2|\omega\rangle\langle\omega|)|s\rangle = |s\rangle - 2|\omega\rangle\langle\omega|s\rangle = |s\rangle - \frac{2}{\sqrt{N}}|\omega\rangle
\end{aligned}$$

$$
\begin{aligned}
U_s\left(|s\rangle - \frac{2}{\sqrt{N}}|\omega\rangle\right) &= \left(2|s\rangle\langle s| - \hat{I}\right)\left(|s\rangle - \frac{2}{\sqrt{N}}|\omega\rangle\right) \\
&= 2|s\rangle\langle s|s\rangle - |s\rangle - \frac{4}{\sqrt{N}}|s\rangle\langle s|\omega\rangle + \frac{2}{\sqrt{N}}|\omega\rangle \\
&= 2|s\rangle - |s\rangle - \frac{4}{\sqrt{N}} \cdot \frac{1}{\sqrt{N}}|s\rangle + \frac{2}{\sqrt{N}}|\omega\rangle = |s\rangle - \frac{4}{N}|s\rangle + \frac{2}{\sqrt{N}}|\omega\rangle \\
&= \frac{N-4}{N}|s\rangle + \frac{2}{\sqrt{N}}|\omega\rangle
\end{aligned}
$$

After the iteration, the probability to measure the marked state has increased from $|\langle\omega|s\rangle|^2 = \frac{1}{N}$ to

$$
|\langle\omega|U_s U_\omega|s\rangle|^2 = \left|\frac{1}{\sqrt{N}} \cdot \frac{N-4}{N} + \frac{2}{\sqrt{N}}\right|^2 = \frac{(3N-4)^2}{N^3} = 9\left(1 - \frac{4}{3N}\right)^2 \cdot \frac{1}{N}.
$$

1. Initialize the system to the state

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle$$

2. Perform the following Grover iteration $r(N)$ times where $r(N)$ is asymptotically $O(\sqrt{N})$:

   a) apply the operator U_ω;
   b) apply the operator U_s.

3. Perform the measurement Ω. The measurement result will be λ_ω with the probability approaching 1 for $N >> 1$. From λ_ω, ω may be obtained.

Supporting material:
Density operators

An operator

$$\rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|$$

is the density operator associated to some ensemble $\{p_i, |\psi_i\rangle\}$ iff it satisfies the conditions:

1. **Trace condition:** $\operatorname{tr} \rho = 1$.
2. **Positivity:** ρ is a positive operator.

Reduced density operator

Suppose we have a physical system A and B whose state is described by the density matrix ρ^{AB}. The reduced density operator for system A is

$$\rho_A = \mathrm{tr}_B\, \rho^{AB}$$

where tr_B is an operator map known as *partial trace* over system B. It is defined as

$$\rho_A = \mathrm{tr}_B\ (|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) = |a_1\rangle\langle a_2| \quad \mathrm{tr}\ (|b_1\rangle\langle b_2|)$$

where $|a_1\rangle$ and $|a_2\rangle$ are any two vectors in A, and $|b_1\rangle$ and $|b_2\rangle$ are any two vectors in B. $\mathrm{tr}\ (|b_1\rangle\langle b_2|)$ is the usual trace, so, using the completeness relation, we get

$$\mathrm{tr}\ (|b_1\rangle\langle b_2|) = \sum_k \langle k|b_1\rangle\langle b_2|k\rangle = \sum_k \langle b_2|k\rangle\langle k|b_1\rangle = \langle b_2|\left(\sum_k |k\rangle\langle k|\right)|b_1\rangle = \langle b_2|b_1\rangle$$

Problem 2: Example

Assume the system is one qubit in the state ρ, and the environment is one qubit in the initial state $|0\rangle$, and the unitary operation is $CNOT$ with the system as the control:

$$
\begin{aligned}
\mathcal{E}(\rho) &= \mathrm{tr}_{env}\left[U_{CNOT}\,(\rho\otimes|0\rangle\langle 0|)\,U_{CNOT}^{\dagger}\right] \\
&= \mathrm{tr}_{env}\left[(P_0\otimes I + P_1\otimes X)\,(\rho\otimes|0\rangle\langle 0|)\,(P_0\otimes I + P_1\otimes X)\right] \\
&= \mathrm{tr}_{env}\left[(P_0\otimes I)\,(\rho\otimes|0\rangle\langle 0|)\,(P_0\otimes I) + (P_0\otimes I)\,(\rho\otimes|0\rangle\langle 0|)\,(P_1\otimes X)\right. \\
&\qquad \left. + (P_1\otimes X)\,(\rho\otimes|0\rangle\langle 0|)\,(P_0\otimes I) + (P_1\otimes X)\,(\rho\otimes|0\rangle\langle 0|)\,(P_1\otimes X)\right] \\
&= \mathrm{tr}_{env}\left[P_0\rho P_0\otimes|0\rangle\langle 0| + P_0\rho P_1\otimes|0\rangle\langle 0|X + P_1\rho P_0\otimes X|0\rangle\langle 0| + P_1\rho P_1\otimes X|0\rangle\langle 0|X\right] \\
&= \mathrm{tr}_{env}\left[P_0\rho P_0\otimes|0\rangle\langle 0| + P_0\rho P_1\otimes|0\rangle\langle 1| + P_1\rho P_0\otimes|1\rangle\langle 0| + P_1\rho P_1\otimes|1\rangle\langle 1|\right] \\
&= P_0\rho P_0\,\langle 0|0\rangle + P_0\rho P_1\,\langle 1|0\rangle + P_1\rho P_0\,\langle 0|1\rangle + P_1\rho P_1\,\langle 1|1\rangle \\
&= P_0\,\rho\,P_0 + P_1\,\rho\,P_1
\end{aligned}
$$