**Assignment 4: selected solutions**

Problem 1:

Consider the following operators $\hat{A} = \hat{Z} \otimes \hat{I}$, $\hat{B} = \hat{X} \otimes \hat{I}$, $\hat{C} = -\frac{1}{\sqrt{2}}\, \hat{I} \otimes \left(\hat{Z} + \hat{X}\right)$ and $\hat{D} = \frac{1}{\sqrt{2}}\, \hat{I} \otimes \left(\hat{Z} - \hat{X}\right)$. Show that the expectation value

$$\langle AC \rangle + \langle BC \rangle + \langle BD \rangle - \langle AD \rangle$$

for a system in the state $|\beta_{11}\rangle = \frac{1}{\sqrt{2}} \left(|01\rangle - |10\rangle\right)$ violates the Bell inequality.

Solution: evaluation of the expectation values:

$$
\begin{aligned}
\langle AC\rangle &= \langle\psi|\frac{-\hat{Z}_1\otimes\hat{Z}_2-\hat{Z}_1\otimes\hat{X}_2}{\sqrt{2}}|\psi\rangle = \frac{1}{2\sqrt{2}}((\langle 01|-\langle 10|)(-\hat{Z}_1\otimes\hat{Z}_2-\hat{Z}_1\otimes\hat{X}_2)((|01\rangle-|10\rangle)) \\
&= \frac{1}{2\sqrt{2}}\Big(-\langle 01|\hat{Z}_1\otimes\hat{Z}_2|01\rangle-\langle 01|\hat{Z}_1\otimes\hat{X}_2|01\rangle+\langle 01|\hat{Z}_1\otimes\hat{Z}_2|10\rangle+\langle 01|\hat{Z}_1\otimes\hat{X}_2|10\rangle \\
&\quad +\langle 10|\hat{Z}_1\otimes\hat{Z}_2|01\rangle+\langle 10|\hat{Z}_1\otimes\hat{X}_2|01\rangle-\langle 10|\hat{Z}_1\otimes\hat{Z}_2|10\rangle-\langle 10|\hat{Z}_1\otimes\hat{X}_2|10\rangle\Big) \\
&= \frac{1}{2\sqrt{2}}(\langle 01|01\rangle-\langle 01|00\rangle-\langle 01|10\rangle-\langle 01|11\rangle-\langle 10|01\rangle+\langle 10|00\rangle+\langle 10|10\rangle+\langle 10|11\rangle) \\
&= \frac{1}{2\sqrt{2}}(1+1)=\frac{1}{\sqrt{2}}
\end{aligned}
$$

Similarly:

$$\begin{aligned}
\langle BC \rangle &= \langle\psi|\frac{-\hat{X}_1 \otimes \hat{Z}_2 - \hat{X}_1 \otimes \hat{X}_2}{\sqrt{2}}|\psi\rangle = \frac{1}{\sqrt{2}}, \\
\langle BD \rangle &= \langle\psi|\frac{\hat{X}_1 \otimes \hat{Z}_2 - \hat{X}_1 \otimes \hat{X}_2}{\sqrt{2}}|\psi\rangle = \frac{1}{\sqrt{2}}, \\
\langle AD \rangle &= \langle\psi|\frac{\hat{Z}_1 \otimes \hat{Z}_2 - \hat{Z}_1 \otimes \hat{X}_2}{\sqrt{2}}|\psi\rangle = -\frac{1}{\sqrt{2}}.
\end{aligned}$$

The final result

$$\langle AC \rangle + \langle BC \rangle + \langle BD \rangle - \langle AD \rangle = \frac{4}{\sqrt{2}} = 2\sqrt{2} > 2$$

violates the Bell inequality which requires that in the classical case the correlation above is bounded by $2$.

Problem 2:

Alice sends you one of the following states $|\psi_1\rangle = |1\rangle$ and $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Show which of the following POVM elements maximize the distinguishability of the states in the measurement outcome:

$$(i)\quad \begin{aligned} E_1 &= \frac{\sqrt{2}}{1+\sqrt{2}}|0\rangle\langle 0|, \\ E_2 &= \frac{\sqrt{2}}{1+\sqrt{2}}\frac{(|0\rangle+|1\rangle)(\langle 0|+\langle 1|)}{2}, \\ E_3 &= 1 - E_1 - E_2, \end{aligned} \qquad (ii)\quad \begin{aligned} E_1 &= \frac{\sqrt{2}}{1+\sqrt{2}}|1\rangle\langle 1|, \\ E_2 &= \frac{\sqrt{2}}{1+\sqrt{2}}\frac{(|0\rangle-|1\rangle)(\langle 0|-\langle 1|)}{2}, \\ E_3 &= 1 - E_1 - E_2. \end{aligned}$$

Solution:

The idea is to distinguish the states by the measurement operators $E_1$ and $E_2$, that is, one of the states can only be detected by $E_1$ and the other by $E_2$.

The case (i):

$$
\begin{aligned}
E_1|\psi_1\rangle &= \frac{\sqrt{2}}{1+\sqrt{2}}|0\rangle\langle 0|1\rangle = 0 \\
E_2|\psi_1\rangle &= \frac{\sqrt{2}}{1+\sqrt{2}}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)|1\rangle = \frac{\sqrt{2}}{2(1+\sqrt{2})}(|0\rangle + |1\rangle) \neq 0 \\
E_1|\psi_2\rangle &= \frac{\sqrt{2}}{\sqrt{2}(1+\sqrt{2})}|0\rangle\langle 0|(|0\rangle - |1\rangle) = \frac{\sqrt{2}}{\sqrt{2}(1+\sqrt{2})}|0\rangle \neq 0 \\
E_2|\psi_2\rangle &= \frac{\sqrt{2}}{\sqrt{2}(1+\sqrt{2})}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)(|0\rangle - |1\rangle) = \frac{\sqrt{2}}{\sqrt{2}(1+\sqrt{2})}(|0\rangle + |1\rangle)(1-1) = 0
\end{aligned}
$$

In the case (i), the states are distinguished by the operators $E_1$ and $E_2$.

In the case (ii) $E_1|\psi_1\rangle \neq 0$, $E_1|\psi_2\rangle \neq 0$, and $E_2|\psi_1\rangle \neq 0$, $E_2|\psi_2\rangle \neq 0$, and the states are not distinguished by the measurement operators in this case.

Problem 3:

Given the Bell state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$,

(i) define the measurement operator for measuring 0 on the first qubit,

(ii) calculate the relevant probability,

(iii) determine the final state immediately after the measurement, and

(iv) given the measurement of the first qubit gave the result 0, calculate the probability of getting the result 0 and 1 for a subsequent measurement on the second qubit.

Solution: (i) Measurement operator for measuring 0 on the first qubit

$$\hat{P}_0^{(1)} = \hat{P}_0 \otimes \hat{I} = |0\rangle\langle 0| \otimes \hat{I} = |0\rangle\langle 0| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) = |00\rangle\langle 00| + |01\rangle\langle 01|.$$

(ii) The probability of measuring 0 on the first qubit

$$p_0^{(1)} = \langle\beta_{00}|\hat{P}_0^{(1)}|\beta_{00}\rangle = \frac{1}{2}(\langle 00| + \langle 11|)\ (|00\rangle\langle 00| + |01\rangle\langle 01|)\ (|00\rangle + |11\rangle) = \frac{1}{2}\langle 00|00\rangle\langle 00|00\rangle = \frac{1}{2}.$$

(iii) The final state after the measurement

$$|\psi\rangle = \frac{\hat{P}_0^{(1)}|\beta_{00}\rangle}{\sqrt{\langle\beta_{00}|\hat{P}_0^{(1)}|\beta_{00}\rangle}} = \frac{\frac{1}{\sqrt{2}}(|00\rangle\langle 00| + |01\rangle\langle 01|)(|00\rangle + |11\rangle)}{\sqrt{\frac{1}{2}}} = |00\rangle.$$

(iv) given the measurement of the first qubit gave the result 0, calculate the probability of getting the result 0 and 1 for a subsequent measurement on the second qubit: the final state after the first measurement is $|00\rangle$, and the measurement operators are $\hat{I}\otimes\hat{P}_0$ and $\hat{I}\otimes\hat{P}_1$. The corresponding probabilities are

$$\begin{aligned} p_0^{(1)} &= \langle 00|\hat{I}\otimes\hat{P}_0|00\rangle = \langle 00|(|00\rangle\langle 00| + |10\rangle\langle 10|)|00\rangle = 1 \\ p_1^{(1)} &= \langle 00|\hat{I}\otimes\hat{P}_1|00\rangle = \langle 00|(|01\rangle\langle 01| + |11\rangle\langle 11|)|00\rangle = 0. \end{aligned}$$

Problem 4:

Given the state $\hat{\rho} = \frac{1}{2}[|00\rangle\langle 00| + |00\rangle\langle 10| + |10\rangle\langle 00| + |10\rangle\langle 10|]$,

(i) define the measurement operator for measuring $0$ on the first qubit,

(ii) calculate the relevant probability,

(iii) determine the final state immediately after the measurement, and

(iv) given the measurement of the first qubit gave the result 0, calculate the probability of getting the result 0 and 1 for a subsequent measurement on the second qubit.

Solution: (i) Measurement operator for measuring 0 on the first qubit

$$\hat{P}_0^{(1)} = \hat{P}_0 \otimes \hat{I} = |0\rangle\langle 0| \otimes \hat{I} = |0\rangle\langle 0| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) = |00\rangle\langle 00| + |01\rangle\langle 01|.$$

(ii) The probability of measuring 0 on the first qubit

$$
\begin{aligned}
p_0^{(1)} &= \mathrm{tr}\left(\hat{P}_0^{(1)}\rho\hat{P}_0^{(1)}\right) = \frac{1}{2}\mathrm{tr}\,(\hat{P}_0 \otimes \hat{I})[|00\rangle\langle 00| + |00\rangle\langle 10| + |10\rangle\langle 00| + |10\rangle\langle 10|](\hat{P}_0 \otimes \hat{I}) \\
&= \mathrm{tr}\left(\frac{1}{2}|00\rangle\langle 00|\right) = \frac{1}{2}.
\end{aligned}
$$

(iii) The final state after the measurement

$$
\rho_0^{(1)} = \frac{\left(\hat{P}_0^{(1)}\rho\hat{P}_0^{(1)}\right)}{\mathrm{tr}\left(\hat{P}_0^{(1)}\rho\hat{P}_0^{(1)}\right)} = \frac{\frac{1}{2}|00\rangle\langle 00|}{\frac{1}{2}} = |00\rangle\langle 00|.
$$

(iv) The final state after the first measurement is $|00\rangle$, and the measurement operators are $\hat{I} \otimes \hat{P}_0$ and $\hat{I} \otimes \hat{P}_1$. The corresponding probabilities are

$$
\begin{aligned}
p_0^{(2)} &= (\hat{I} \otimes \hat{P}_0)|00\rangle\langle 00|(\hat{I} \otimes \hat{P}_0) = (|00\rangle\langle 00| + |10\rangle\langle 10|)\ |00\rangle\langle 00|\ (|00\rangle\langle 00| + |10\rangle\langle 10|) = 1 \\
p_1^{(2)} &= (\hat{I} \otimes \hat{P}_1)|00\rangle\langle 00|(\hat{I} \otimes \hat{P}_1) = (|01\rangle\langle 01| + |11\rangle\langle 11|)\ |00\rangle\langle 00|\ (|01\rangle\langle 01| + |11\rangle\langle 11|) = 0.
\end{aligned}
$$

**Quantum search algorithm** (Grover)

Consider an unsorted database with $N = 2^n$ entries where $n$ is the number of qubits. The problem is to determine the index of the database entry which satisfies some search criterion, that is, to identify the marked state $|\omega\rangle$.

We are provided with oracle access to a unitary operator, $U_\omega$, which acts as follows:

$$\begin{aligned} U_\omega|\omega\rangle &= -|\omega\rangle \\ U_\omega|x\rangle &= |x\rangle, \text{ for all } x \neq \omega. \end{aligned}$$

The operator $U_\omega$ can be rewritten as

$$U_\omega = \hat{I} - 2|\omega\rangle\langle\omega|$$

$$\begin{aligned} (\hat{I} - 2|\omega\rangle\langle\omega|)\,|\omega\rangle &= |\omega\rangle - 2|\omega\rangle\langle\omega|\omega\rangle = -|\omega\rangle, \\ (\hat{I} - 2|\omega\rangle\langle\omega|)\,|x\rangle &= |x\rangle - |\omega\rangle\langle\omega|x\rangle = |x\rangle. \end{aligned}$$

Let $|s\rangle$ denote the uniform superposition over all states

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle$$

We introduce the Grover diffusion operator

$$U_s = 2|s\rangle\langle s| - \hat{I}.$$

The following computations show what happens in the first iteration:

$$\begin{aligned}
\langle s|\omega\rangle &= \frac{1}{\sqrt{N}} \\
\langle s|s\rangle &= N\frac{1}{\sqrt{N}} \cdot \frac{1}{\sqrt{N}} = 1 \\
U_\omega|s\rangle &= (\hat{I} - 2|\omega\rangle\langle\omega|)|s\rangle = |s\rangle - 2|\omega\rangle\langle\omega|s\rangle = |s\rangle - \frac{2}{\sqrt{N}}|\omega\rangle
\end{aligned}$$

$$
\begin{aligned}
U_s\left(|s\rangle - \frac{2}{\sqrt{N}}|\omega\rangle\right) &= \left(2|s\rangle\langle s| - \hat{I}\right)\left(|s\rangle - \frac{2}{\sqrt{N}}|\omega\rangle\right) \\
&= 2|s\rangle\langle s|s\rangle - |s\rangle - \frac{4}{\sqrt{N}}|s\rangle\langle s|\omega\rangle + \frac{2}{\sqrt{N}}|\omega\rangle \\
&= 2|s\rangle - |s\rangle - \frac{4}{\sqrt{N}} \cdot \frac{1}{\sqrt{N}}|s\rangle + \frac{2}{\sqrt{N}}|\omega\rangle = |s\rangle - \frac{4}{N}|s\rangle + \frac{2}{\sqrt{N}}|\omega\rangle \\
&= \frac{N-4}{N}|s\rangle + \frac{2}{\sqrt{N}}|\omega\rangle
\end{aligned}
$$

After the iteration, the probability to measure the marked state has increased from $|\langle\omega|s\rangle|^2 = \frac{1}{N}$ to

$$
|\langle\omega|U_s U_\omega|s\rangle|^2 = \left|\frac{1}{\sqrt{N}} \cdot \frac{N-4}{N} + \frac{2}{\sqrt{N}}\right|^2 = \frac{(3N-4)^2}{N^3} = 9\left(1 - \frac{4}{3N}\right)^2 \cdot \frac{1}{N}.
$$

1. Initialize the system to the state

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle$$

2. Perform the following Grover iteration $r(N)$ times where $r(N)$ is asymptotically $O(\sqrt{N})$:

   a) apply the operator $U_\omega$;
   b) apply the operator $U_s$.

3. Perform the measurement $\Omega$. The measurement result will be $\lambda_\omega$ with the probability approaching $1$ for $N >> 1$. From $\lambda_\omega$, $\omega$ may be obtained.