Assignment 2: selected solutions

I. Quantum circuits and protocols

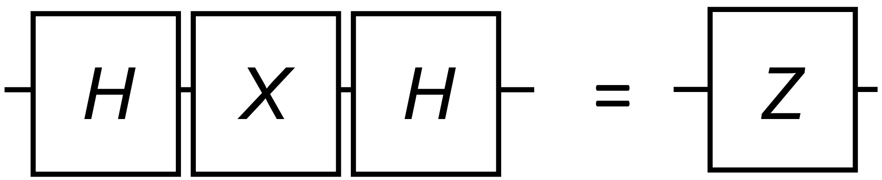
Problem 1:

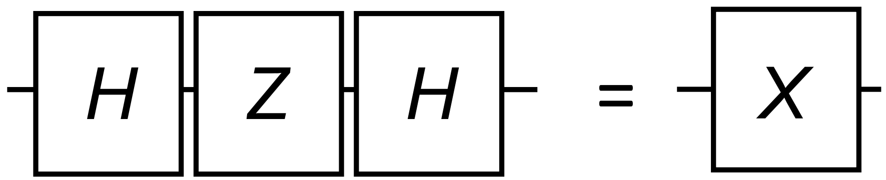
Determine whether the Hadamard gate is self-adjoint, unitary or projection operator, and show the net result of the following sequences of the single qubit gates: (i) HXH, (ii) HZH.

Solution: The Hadamard gate H is self-adjoint and unitary: $H = H^{\dagger}$, $HH = \hat{1}$.

$$HXH = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z,$$

$$HZH = HHXHH = X.$$



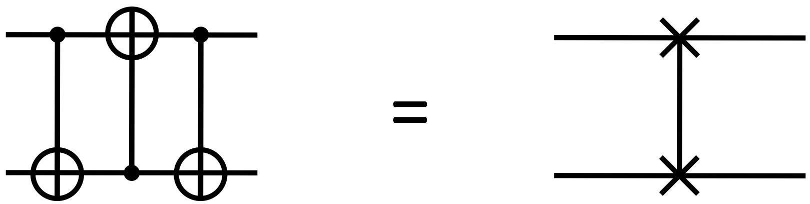


Problem 2:

Calculate to what matrix in the standard computational representation the circuits $(CNOT_{12})(CNOT_{21})(CNOT_{12})$ corresponds.

Solution:

$$(CNOT_{12})(CNOT_{21})(CNOT_{12}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = SWAP$$



Problem 3:
Verify in the standard matrix representation that

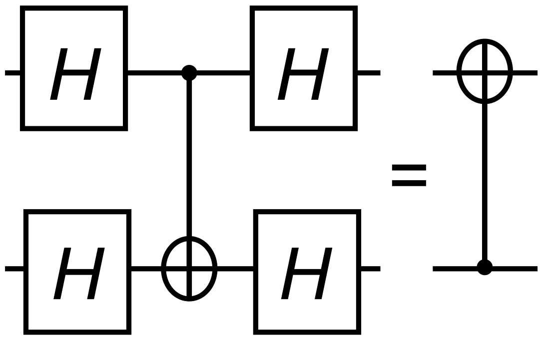
$$CNOT_{21} = (H_1 \otimes H_2)(CNOT_{12})(H_1 \otimes H_2).$$

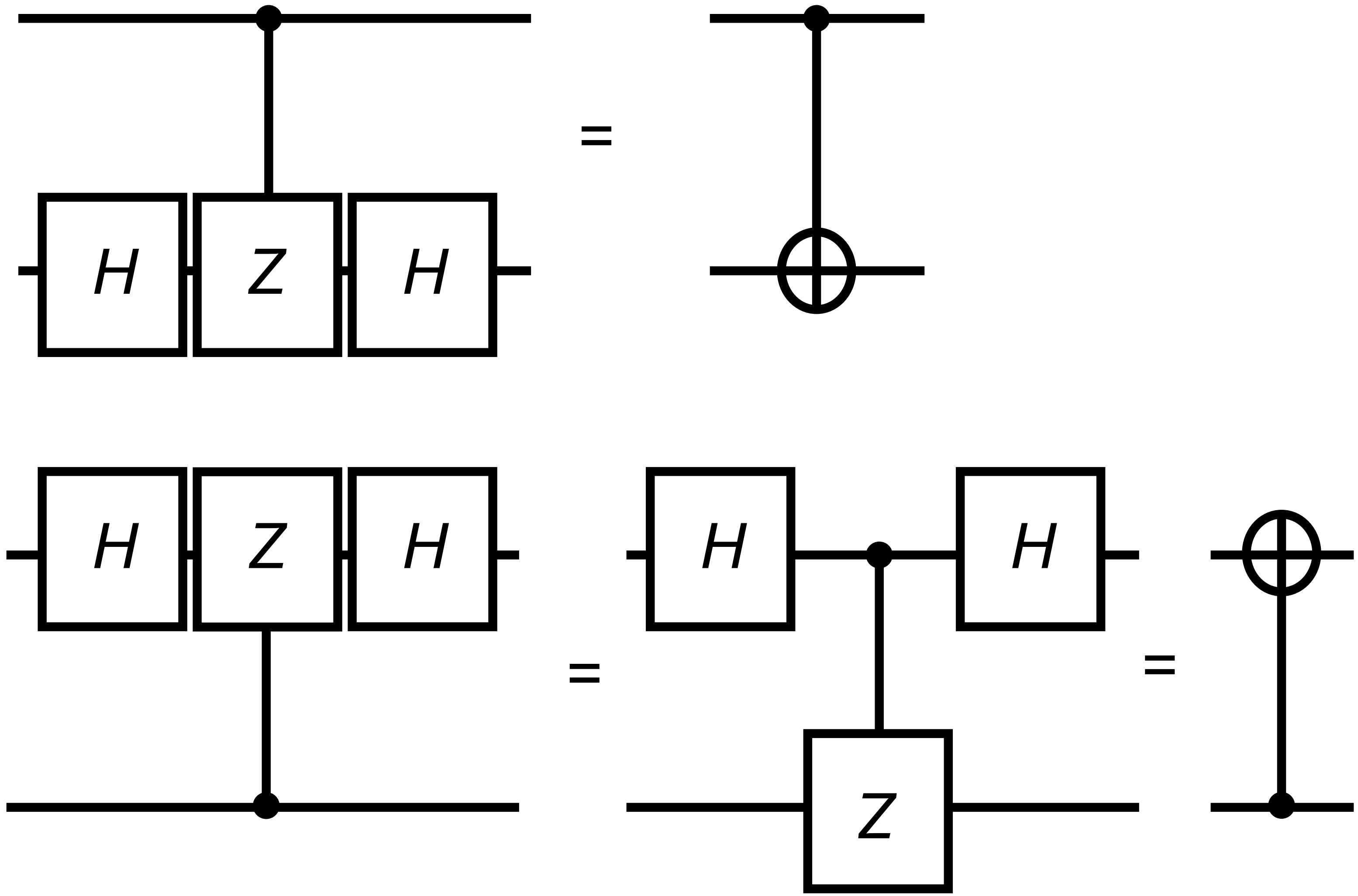
Draw the circuit as a quantum circuit diagram.

Solution:

$$(H_1 \otimes H_2)(CNOT_{12})(H_1 \otimes H_2) = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = CNOT_{21}.$$

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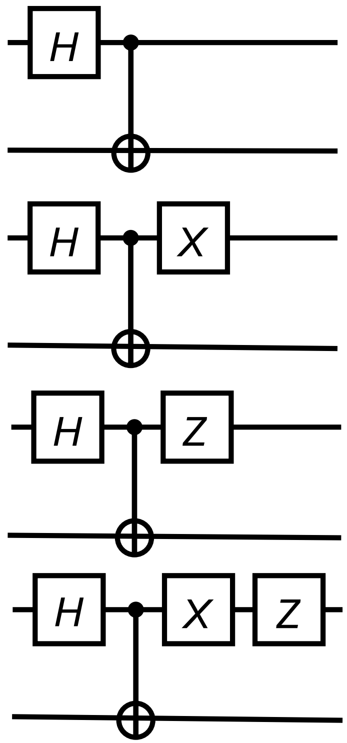
Problem 4:

Construct the circuits to generate all Bell states from the fiducial initial two-qubit state $|00\rangle$ and construct a quantum circuit that transforms the Bell states into the standard computational basis states.

Solution:

$$
\begin{aligned}
|\beta_{00}\rangle &= \frac{|00\rangle + |11\rangle}{\sqrt{2}} \qquad (CNOT_{12})(H_1) \\
|\beta_{01}\rangle &= \frac{|01\rangle + |10\rangle}{\sqrt{2}} \qquad (X_1)(CNOT_{12})(H_1) \\
|\beta_{10}\rangle &= \frac{|00\rangle - |11\rangle}{\sqrt{2}} \qquad (Z_1)(CNOT_{12})(H_1) \\
|\beta_{11}\rangle &= \frac{|01\rangle - |10\rangle}{\sqrt{2}} \qquad (Z_1)(X_1)(CNOT_{12})(H_1)
\end{aligned}
$$

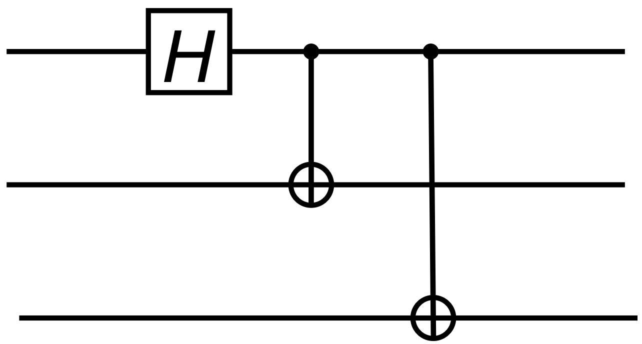


Problem 5:

Formulate a circuit to generate the Greenberger-Horne-Zeilinger (GHZ) state $|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$.

Solution:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \qquad (CNOT_{13})(CNOT_{12})(H_1)$$



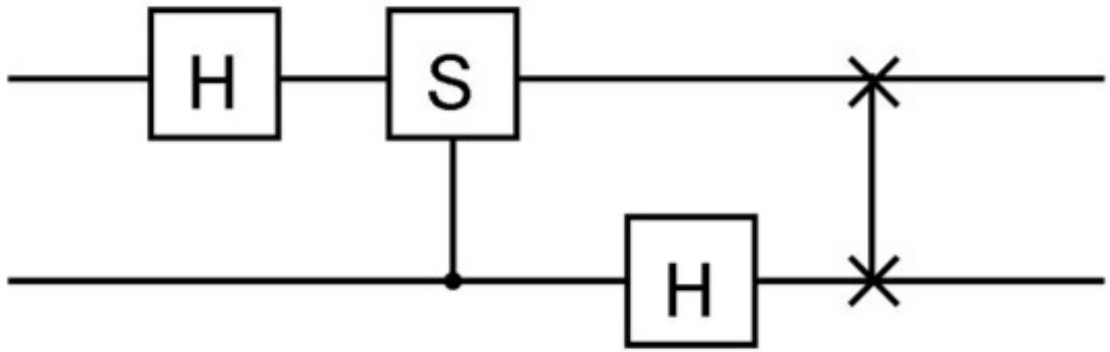
Problem 6:

Write down the circuit $(SWAP)(H_2)(CS_{21})(H_1)$ in the standard matrix representation.

Draw the circuit as a quantum circuit diagram.

Solution:

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$



II. Quantum states

Problem 4:
Determine whether the state $|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$ is separable or entangled using the Schmidt decomposition.

Solution:
$a_{00} = \frac{1}{2}, a_{01} = \frac{1}{2}, a_{10} = \frac{1}{2}, a_{11} = -\frac{1}{2}$

$$aa^{\dagger} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

This matrix is already diagonal. There are two non-zero Schmidt coefficients and thus the Schmidt number is 2, and the state is entangled.

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Any 2×2 matrix can be written as a linear combination of the matrices $\{I, \sigma_x, \sigma_y, \sigma_z\}$.

Properties:

The Pauli matrices are both hermitian and unitary $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \hat{1}$, and their products are

$$\sigma_x\sigma_y = i\sigma_z, \qquad \sigma_y\sigma_z = i\sigma_x, \qquad \sigma_z\sigma_x = i\sigma_y.$$

Notice that for $\alpha \neq \beta$, where $\alpha, \beta = x, y, z$, they satisfy

$$\sigma_\alpha\sigma_\beta\sigma_\alpha = -\sigma_\beta.$$

Pauli matrices anti-commute

$$\{\sigma_x, \sigma_y\} = \sigma_x\sigma_y + \sigma_y\sigma_x = \{\sigma_y, \sigma_z\} = \{\sigma_z, \sigma_x\} = 0$$

and satisfy the commutation relations

$$\left[\sigma_x, \sigma_y\right] = 2i\sigma_z, \qquad \left[\sigma_y, \sigma_z\right] = 2i\sigma_x, \qquad [\sigma_z, \sigma_x] = 2i\sigma_y.$$

They are traceless and their determinant is -1

$$\begin{aligned} \operatorname{tr}\sigma_x = \operatorname{tr}\sigma_y = \operatorname{tr}\sigma_z &= 0 \\ \det\sigma_x = \det\sigma_y = \det\sigma_x &= -1 \end{aligned}$$

$$\left(\vec{\sigma}\cdot\vec{A}\right)\left(\vec{\sigma}\cdot\vec{B}\right) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot\left(\vec{A}\times\vec{B}\right)$$

Taylor expansion of single-qubit evolution operators

Assignment 3, Problem 3

$$
\begin{aligned}
\hat{R}_{\vec{n}}(\theta) &= e^{-i\theta\ \vec{n}\cdot\vec{\sigma}\ /2} \\
&= \sum_k \frac{(-i\theta\ \vec{n}\cdot\vec{\sigma}\ /2)^k}{k!} \\
&= \sum_k (-i)^k \frac{(\theta/2)^k}{k!}\ (\vec{k}\cdot\vec{\sigma})^k
\end{aligned}
$$

To complete the solution will require:

- properties of Pauli matrices, and
- Taylor expansions of trigonometric functions.

Lie groups $U(4)$ and $SU(4)$

$U(4)$ is the group of unitary 4×4 matrices or operators, and $SU(4)$ is the group of unitary 4×4 matrices of the unit determinant. Elements $u \in U(4)$ can be expressed in terms of elements $g \in SU(4)$:

$$u = e^{i\alpha} g = \left(e^{i\alpha}\hat{1}\right)\ g$$

where $\hat{1}$ is a 4×4 unit matrix. Considering the determinant of the product of two $n \times n$ matrices A and B, and using $\det(AB) = \det A \det B$, we get

$$\det u = \det\left(e^{i\alpha} g\right) = \det\left[\left(e^{i\alpha}\hat{1}\right) g\right] = \det\left(e^{i\alpha}\hat{1}\right)\det g = e^{i4\alpha}$$

from which we get in the two-qubit case

$$g = \frac{u}{\sqrt[4]{\det u}}.$$

Local invariants

1) Unitary transformation of any $U \in SU(4)$ into the Bell basis

$$U_B = U_Q^\dagger U U_Q = U_Q^\dagger k_1 U_Q U_Q^\dagger A U_Q U_Q^\dagger k_2 U_Q = O_1 F O_2,$$

$$\text{where } U_Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & i & 1 & 0 \\ 0 & i & -1 & 0 \\ 1 & 0 & 0 & -i \end{pmatrix}$$

2) Constructing the Makhlin matrix

$$m = U_B^T U_B = O_2^T F O_1^T O_1 F O_2 = O_2^T F^2 O_2$$

3) Computing the local invariants

$$g_1 = \frac{1}{16} Re\left[\mathrm{tr}^2(m)\right], \quad g_2 = \frac{1}{16} Im\left[\mathrm{tr}^2(m)\right], \quad g_3 = \frac{1}{4}\left[\mathrm{tr}^2(m) - \mathrm{tr}\,(m^2)\right].$$

For general two-qubit unitary matrices $U \in U(4)$ whose determinants are any complex number of unit modulus, we define the local invariants as

$$g_1 = \frac{Re\left[\mathrm{tr}^2(m)\right]}{16\det(U)}, \quad g_2 = \frac{Im\left[\mathrm{tr}^2(m)\right]}{16\det(U)}, \quad g_3 = \frac{\left[\mathrm{tr}^2(m) - \mathrm{tr}(m^2)\right]}{4\det(U)}.$$

Examples:

1) Identity/unit matrix $\hat{1} \in SU(4)$

$$U_B = U_Q^\dagger \hat{1} U_Q = U_Q^\dagger U_Q = \hat{1}$$

$$m = U_B^T U_B = \hat{1}$$

$$g_1 = \frac{1}{16} Re\left[\mathrm{tr}^2(m)\right] = 1, \quad g_2 = \frac{1}{16} Im\left[\mathrm{tr}^2(m)\right] = 0, \quad g_3 = \frac{1}{4}\left[\mathrm{tr}^2(m) - \mathrm{tr}(m^2)\right] = 3.$$

2) The operation $(H_1 \otimes H_2) \in SU(2) \otimes SU(2) \subset SU(4)$

$$U_B = U_Q^\dagger (H_1 \otimes H_2) U_Q = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -i & -i & 0 \\ 0 & 1 & -1 & 0 \\ -i & 0 & 0 & i \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & i & 1 & 0 \\ 0 & i & -1 & 0 \\ 1 & 0 & 0 & -i \end{pmatrix}$$

$$m = U_B^T U_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$g_1 = \frac{1}{16} Re\left[\mathrm{tr}^2(m)\right] = 1, \quad g_2 = \frac{1}{16} Im\left[\mathrm{tr}^2(m)\right] = 0, \quad g_3 = \frac{1}{4}\left[\mathrm{tr}^2(m) - \mathrm{tr}\,(m^2)\right] = 3.$$

These values of the invariants show that the single-qubit operation $(H_1 \otimes H_2)$ is locally equivalent to the identity operator $\hat{1}$ studied in the previous example. Indeed all the elements of $SU(2) \otimes SU(2)$ are in the local equivalence class of the identity $\left[\hat{1}\right]$.

3) Controlled phase-flip CPHASE

$$U_B = U_Q^\dagger (CPHASE) U_Q = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -i & -i & 0 \\ 0 & 1 & -1 & 0 \\ -i & 0 & 0 & i \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & i \\ 0 & i & 1 & 0 \\ 0 & i & -1 & 0 \\ 1 & 0 & 0 & -i \end{pmatrix}$$

$$m = U_B^T U_B = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Using the local invariants defined for $U(4)$ and $\det(U) = -1$, we get

$$g_1 = \frac{Re\left[\mathrm{tr}^2(m)\right]}{16\det(U)} = 0, \quad g_2 = \frac{Im\left[\mathrm{tr}^2(m)\right]}{16\det(U)} = 0, \quad g_3 = \frac{\left[\mathrm{tr}^2(m) - \mathrm{tr}\,(m^2)\right]}{4\det(U)} = 1.$$