Assignment 1: selected solutions

Problem 1: Show how the gates $Y = -iZX$, $S = \sqrt{Z}$ and $T = \sqrt{S}$ transform a general state of one quantum bit.

Solution: The gates Z and X transform the standard basis vectors as follows:

$$Z|0\rangle = |0\rangle, \qquad Z|1\rangle = -|1\rangle,$$
$$X|0\rangle = |1\rangle, \qquad X|1\rangle = |0\rangle,$$

so the action of the gate $Y = -iZX$ onto a general qubit state $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ as

$$\begin{aligned} Y|\psi\rangle &= -iZX(c_0|0\rangle + c_1|1\rangle) \\ &= -iZ(c_0X|0\rangle + c_1X|1\rangle) \\ &= -iZ(c_1|0\rangle + c_0|1\rangle) \\ &= -i(c_1|0\rangle - c_0|1\rangle). \end{aligned}$$

From the following

$$\begin{aligned} Z|\psi\rangle &= Z(c_0|0\rangle + c_1|1\rangle) \\ &= c_0 Z|0\rangle + c_1 Z|1\rangle \\ &= c_0|0\rangle - c_1|1\rangle \\ &= c_0 S^2|0\rangle + c_1 S^2|1\rangle. \end{aligned}$$

we observe that $S^2|0\rangle = |0\rangle = e^{i0}|0\rangle$ and $S^2|1\rangle = -|1\rangle = e^{i\pi}|1\rangle$. The action of the gate S onto a general state of one quantum bit

$$\begin{aligned} S|\psi\rangle &= S(c_0|0\rangle + c_1|1\rangle) \\ &= c_0 S|0\rangle + c_1 S|1\rangle \\ &= c_0|0\rangle + e^{i\pi/2} c_1|1\rangle \\ &= c_0|0\rangle + i c_1|1\rangle, \end{aligned}$$

and similarly

$$T|\psi\rangle = c_0|0\rangle + e^{i\pi/4} c_1|1\rangle.$$

Single-qubit operations in the standard computational basis:

(i) Phase flip

$$\begin{aligned}
\hat{Z} &= \left(\sum_{k=0,1} |k\rangle\langle k|\right)\hat{Z}\left(\sum_{l=0,1} |l\rangle\langle l|\right) = \sum_{k,l}\langle k|\hat{Z}|l\rangle\ |k\rangle\langle l| \\
&= \langle 0|\hat{Z}|0\rangle|0\rangle\langle 0| + \langle 0|\hat{Z}|1\rangle|0\rangle\langle 1| + \langle 1|\hat{Z}|0\rangle|1\rangle\langle 0| + \langle 1|\hat{Z}|1\rangle|1\rangle\langle 1| \\
&= \langle 0|\hat{Z}|0\rangle\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + \langle 0|\hat{Z}|1\rangle\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} \\
&+ \langle 1|\hat{Z}|0\rangle\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + \langle 1|\hat{Z}|1\rangle\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \langle 0|\hat{Z}|0\rangle & \langle 0|\hat{Z}|1\rangle \\ \langle 1|\hat{Z}|0\rangle & \langle 1|\hat{Z}|1\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}$$

(ii) Bit flip

$$\hat{X} = \begin{pmatrix} \langle 0|\hat{X}|0\rangle & \langle 0|\hat{X}|1\rangle \\ \langle 1|\hat{X}|0\rangle & \langle 1|\hat{X}|1\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Problem 2: Write the gates $Y = -iZX$, $S = \sqrt{Z}$ and $T = \sqrt{S}$ in the Dirac notation and in the matrix representation in the standard computational basis.

Solution: We can for example write the gate Z as $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ which indeed transforms the standard basis states correctly, e.g. $Z|0\rangle = |0\rangle\langle 0|0\rangle - |1\rangle\langle 1|0\rangle = |0\rangle$ and similarly $X = |0\rangle\langle 1| + |1\rangle\langle 0|$.

$$
\begin{aligned}
Y &= -iZX \\
&= -i(|0\rangle\langle 0| - |1\rangle\langle 1|)(|0\rangle\langle 1| + |1\rangle\langle 0|) \\
&= -i|0\rangle\langle 1| + i|1\rangle\langle 0| \\
&= -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\end{aligned}
$$

For the rest of the problem 2 we use also functions of operators. After observing that the gate Z is diagonal in the standard basis, we get

$$S = \sqrt{Z} = \begin{pmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix} = \begin{pmatrix} \sqrt{e^{i0}} & 0 \\ 0 & \sqrt{e^{i\pi}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

Problem 3: Show using the standard matrix representation that you can write the controlled-NOT operations using the projection operators as follows:

$$CNOT_{12} = \hat{P}_0 \otimes \hat{I} + \hat{P}_1 \otimes \hat{X}$$

where $\hat{P}_0 = |0\rangle\langle 0|$ and $\hat{P}_1 = |1\rangle\langle 1|$ are the single-qubit projectors, and $\hat{I}$ is the single-qubit identity, i.e. 2×2 unit matrix, and $\hat{X}$ is the single-qubit bit-flip operator.

Solution: We first have to construct the matrix representation. The gate $CNOT_{12}$ acts on the standard basis elements as $CNOT_{12}|00\rangle = |00\rangle$, $CNOT_{12}|01\rangle = |01\rangle$, $CNOT_{12}|10\rangle = |11\rangle$ and $CNOT_{12}|11\rangle = |10\rangle$, which gives its matrix representation

$$CNOT_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We have to show that the r.h.s. gives the same matrix

$$\hat{P}_0 \otimes \hat{I} + \hat{P}_1 \otimes \hat{X} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Problem 4: Consider the controlled-$PHASE$ gate, that flips the phase of the target qubit if the controlled qubit is in the state $|1\rangle$, and show in the standard matrix representation that $CPHASE_{12} = CPHASE_{21}$.

Solution:

$$CPHASE_{12} = \hat{P}_0 \otimes \hat{I} + \hat{P}_1 \otimes \hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$
\begin{aligned}
CPHASE_{21} &= \hat{I} \otimes \hat{P}_0 + \hat{Z} \otimes \hat{P}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\end{aligned}
$$

which shows that

$$CPHASE_{12} = CPHASE_{21}.$$

Problem 5: Consider the controlled-S gates, that applies the S gate to the target qubit if the controlled qubit is in the state $|1\rangle$. Write down the controlled-S gates CS_{12} and CS_{21} (i) in the standard matrix representation and (ii) in compact form using appropriate projectors.

Solution:

$$
\begin{aligned}
CS_{12} &= \hat{P}_0 \otimes \hat{I} + \hat{P}_1 \otimes \hat{S} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}
\end{aligned}
$$

and similarly

$$CS_{21} = \hat{I} \otimes \hat{P}_0 + \hat{S} \otimes \hat{P}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}.$$

Assignment 2: supporting material

Types of operators

1. The hermitian conjugate or adjoint of a matrix is given as $\hat{A}^\dagger = (\hat{A}^T)^*$.

2. An operator $\hat{A}$ is called **hermitian** if $\hat{A}^\dagger = \hat{A}$, or $\langle \hat{A}\phi|\psi\rangle = \langle\phi|\hat{A}\psi\rangle$.

3. An operator $\hat{U}$ is called **unitary** if $\hat{U}^\dagger = \hat{U}^{-1}$, that is $\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \hat{1}$.

4. An operator $\hat{P}$ satisfying $\hat{P} = \hat{P}^\dagger$ and $\hat{P} = \hat{P}^2$ is a **projection operator** or **projector**.

Example:

1. $CNOT^{\dagger}_{12} = (CNOT^{T}_{12})^{*} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

2. $CNOT_{12} = CNOT^{\dagger}_{12}$ so the operator is hermitian. The matrix representation of $CNOT_{12}$ is real and symmetric. .

3. $(CNOT_{12})(CNOT_{12})^{\dagger} = (CNOT_{12})^{\dagger}(CNOT_{12}) = (CNOT_{12})(CNOT_{12}) = \hat{I}$, so the operation is unitary.

4. $CNOT_{12}$ is hermitian but it is not idempotent $(CNOT_{12})(CNOT_{12}) \neq (CNOT_{12})$, so it is not a projector.

Density operator/matrix

We can represent a qubit state $|\phi\rangle$, and any quantum state, by the projector onto the one-dimensional subspace it spans:

$$\begin{aligned}\hat{\rho} &= |\phi\rangle\langle\phi| = (c_0|0\rangle + c_1|1\rangle)\left(c_0^*\langle 0| + c_1^*\langle 1|\right) \\ &= \left|c_0\right|^2 \; |0\rangle\langle 0| + c_0 c_1^* \; |0\rangle\langle 1| + c_0^* c_1 \; |1\rangle\langle 0| + |c_1|^2 \; |1\rangle\langle 1|\end{aligned}$$

In matrix representation given by the standard computational basis, we have

$$\hat{\rho} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \begin{pmatrix} c_0^* & c_1^* \end{pmatrix} = \begin{pmatrix} \left|c_0\right|^2 & c_0 c_1^* \\ c_0^* c_1 & |c_1|^2 \end{pmatrix}$$

We observe that the **norm** of a state is $\mathrm{Tr}(\hat{\rho}) = |c_0|^2 + |c_1|^2 = 1$ and also that $\rho_{10} = \rho_{01}^*$.

Bloch representation

The single-qubit density matrix can be decomposed as follows

$$\hat{\rho} = \frac{1}{2}\left(\hat{I} + \vec{r}\,.\,\vec{\sigma}\right) = \frac{1}{2}\left(\hat{I} + r_x\,\sigma_x + r_y\,\sigma_y + r_z\,\sigma_z\right)$$

$$= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + r_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + r_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + r_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix}$$

where σ_x, σ_y, and σ_z are the Pauli matrices and the vector $\vec{r} = \left(r_x, r_y, r_z\right)$ is called the **Bloch vector**.

The Bloch vector components are real numbers between 0 between 1, are related to the density matrix elements as shown (for a pure state)

$$
\begin{aligned}
r_x &= 2\,\mathrm{Re}\,(\rho_{10}) &= 2\,\mathrm{Re}\left(c_0^* c_1\right) \\
r_y &= 2\,\mathrm{Im}\,(\rho_{10}) &= 2\,\mathrm{Im}\left(c_0^* c_1\right) \quad . \\
r_z &= \rho_{00} - \rho_{11} &= \left|c_0\right|^2 - |c_1|^2
\end{aligned}
$$

1. $|\phi\rangle = |0\rangle$

$$\hat{\rho} = |0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1^* & 0^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \vec{r} = (0, 0, 1)$$

2. $|\phi\rangle = |1\rangle$

$$\hat{\rho} = |1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0^* & 1^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \vec{r} = (0, 0, -1)$$

3. $|\phi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$

$$\hat{\rho} = |\phi\rangle\langle \phi| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}^* & \frac{1}{\sqrt{2}}^* \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ \\ 1 & 1 \end{pmatrix} \quad \Rightarrow \quad \vec{r} = (1, 0, 0)$$

4. $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$$\hat{\rho} = |\phi\rangle\langle\phi| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}^* & -\frac{1}{\sqrt{2}}^* \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ \\ -1 & 1 \end{pmatrix} \quad \Rightarrow \quad \vec{r} = (-1, 0, 0)$$

5. $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

$$\hat{\rho} = |\phi\rangle\langle\phi| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \\ i\,\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}^* & \left(i\,\frac{1}{\sqrt{2}}\right)^* \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ \\ i & 1 \end{pmatrix} \quad \Rightarrow \quad \vec{r} = (0, 1, 0)$$