QUANTUM MECHANICS FOUNDATIONS OF QUANTUM INFORMATION PROCESSING

MEASUREMENT

FOURTH POSTULATE
(Measurement I)

The only possible result of the measurement of a physical quantity $\mathscr{A}$ is one of the eigenvalues of the corresponding observable $\hat{A}$.

FIFTH POSTULATE
(Measurement II)

1. a discrete non-degenerate spectrum:
   When the physical quantity $\mathscr{A}$ is measured on a system in the normalized state $|\psi\rangle$, the probability $\mathcal{P}(a_n)$ of obtaining the non-degenerate eigenvalue a_n of the corresponding physical observable $\hat{A}$ is

$$\mathcal{P}(a_n) = |\langle u_n|\psi\rangle|^2$$

   where $|u_n\rangle$ is the normalised eigenvector of $\hat{A}$ associated with the eigenvalue a_n.

2. a discrete spectrum:

$$\mathcal{P}(a_n) = \sum_{i=1}^{g_n} \left|\langle u_n^i|\psi\rangle\right|^2$$

John von Neumann, Mathematical Foundations of Quantum Mechanics where g_n is the degree of degeneracy of a_n and $\{|u_n^i\rangle\}$ $(i = 1, \ldots, g_n)$ is an orthonormal set of vectors which forms a basis in the eigenspace $\mathcal{H}_n$ associated with the eigenvalue a_n of the observable $\hat{A}$.

3. a continuous spectrum:
   the probability $\mathrm{d}\mathcal{P}(\alpha)$ of obtaining result included between α and $\alpha + \mathrm{d}\alpha$ is

$$\mathrm{d}\mathcal{P}(\alpha) = |\langle v_\alpha|\psi\rangle|^2 \, \mathrm{d}\alpha$$

   where $|v_\alpha\rangle$ is the eigenvector corresponding to the eigenvalue α of the observable $\hat{A}$.

SIXTH POSTULATE
(Measurement III)

If the measurement of the physical quantity $\mathcal{A}$ on the system in the state $|\psi\rangle$ gives the result a_n, the state of the system immediately after the measurement is the mor-malized projection

$$\frac{\hat{P}_n|\psi\rangle}{\sqrt{\langle\psi|\hat{P}_n|\psi\rangle}} = \frac{\hat{P}_n|\psi\rangle}{\left\|\hat{P}_n|\psi\rangle\right\|}$$

of $|\psi\rangle$ onto the eigensubspace associated with a_n.

John von Neumann, Mathematical Foundations of Quantum Mechanics

General measurement

Measurement is defined by the set of measurement operators $\{\hat{M}_m\}$ where m refers to the measurement outcomes.

If the state of the system before the measurement is $|\phi\rangle$, then the probability that result m occurs is

$$p_m = \langle\phi|\hat{M}_m^{\dagger}\hat{M}_m|\phi\rangle$$

and the state after the measurement is

$$|\psi\rangle = \frac{\hat{M}_m|\phi\rangle}{||\hat{M}_m|\phi\rangle||} = \frac{\hat{M}_m|\phi\rangle}{\sqrt{\langle\phi|\hat{M}_m^{\dagger}\hat{M}_m|\phi\rangle}}$$

The measurement operators satisfy the completeness relation

$$\sum_m \hat{M}_m^\dagger \hat{M}_m = \hat{I}$$

which expresses the fact that the probabilities of of measurement results sum to unity

$$\sum_m \langle\phi|\hat{M}_m^\dagger \hat{M}_m|\phi\rangle = \sum_m p_m = 1$$

Distinguishing quantum states

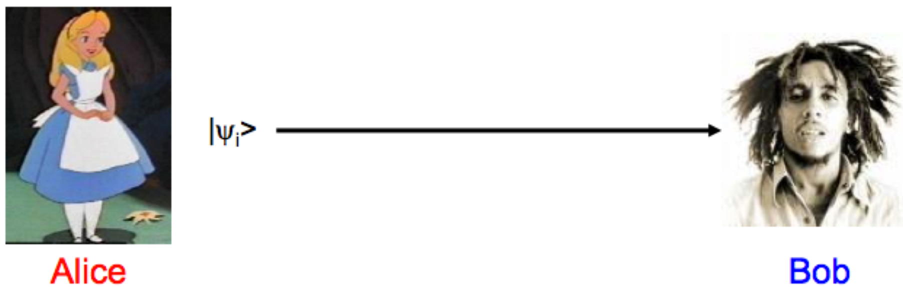
Two-parties game: Alice chooses a state $|\psi_i\rangle$, where $1 \leq i \leq n$, from some fixed set of states known to both parties, and sends it to Bob whose task is to identify it.

If the states $\{|\psi_i\rangle\}$ are **orthogonal** than Bob can perform a quantum measurement to distinguish the states: Bob has to define the measurement operators

$$\begin{aligned} \hat{M}_i &= |\psi_i\rangle\langle\psi_i| \\ \hat{M}_0 &= \sqrt{\hat{I} - \sum_{i \neq 0} |\psi_i\rangle\langle\psi_i|} \qquad \text{(positive square-root)} \end{aligned}$$

which satisfy the completeness relation and thus can be used to distinguish the state.

If the states $\{|\psi_i\rangle\}$ are **non-orthogonal** than there is no quantum measurement to reliably distinguish the states.



Projective measurement

A projective measurement is described by an observable $\hat{M}$, a self-adjoint operator on a state space of the system which is being observed. The observable has the spectral decomposition

$$\hat{M} = \sum_m \lambda_m \hat{P}_m$$

where $\hat{P}_m$ is the projector onto the eigenspace of $\hat{M}$ associated with the eigenvalue λ_m.

The possible outcomes of the measurement correspond to the eigenvalues λ_m of the observable $\hat{M}$.

If the state of the system before the measurement is $|\phi\rangle$, then the probability that the result λ_m occurs is

$$p_m = \langle\phi|\hat{P}_m^\dagger \hat{P}_m|\phi\rangle = \langle\phi|\hat{P}_m^2|\phi\rangle = \langle\phi|\hat{P}_m|\phi\rangle$$

and the state immediately after the measurement is

$$|\psi\rangle = \frac{\hat{P}_m|\phi\rangle}{\|\hat{P}_m|\phi\rangle\|} = \frac{\hat{P}_m|\phi\rangle}{\sqrt{\langle\phi|\hat{P}_m^\dagger \hat{P}_m|\phi\rangle}} = \frac{\hat{P}_m|\phi\rangle}{\sqrt{\langle\phi|\hat{P}_m|\phi\rangle}} = \frac{\hat{P}_m|\phi\rangle}{\sqrt{p_m}}$$

Projective measurement allows us to easily calculate the expectation value of an observable $\hat{M}$ for the system in the state $|\phi\rangle$

$$< \hat{M} >= \langle\phi|\hat{M}|\phi\rangle = \langle\phi|\left(\sum_m \lambda_m \hat{P}_m\right)|\phi\rangle = \sum_m \lambda_m \langle\phi|\hat{P}_m|\phi\rangle = \sum_m \lambda_m p_m$$

Heisenberg uncertainty relation

Let $\hat{A}$ and $\hat{B}$ be self-adjoint operators, and $|\phi\rangle$ be a quantum state.
Suppose $\langle\phi|\hat{A}\hat{B}|\phi\rangle = x + iy$, where $x, y \in \mathbb{R}$ and note that

$$\begin{aligned}\langle\phi|\left[\hat{A},\hat{B}\right]|\phi\rangle &= 2iy\\ \langle\phi|\left\{\hat{A},\hat{B}\right\}|\phi\rangle &= 2x\end{aligned}$$

This implies

$$\left|\langle\phi|\left[\hat{A},\hat{B}\right]|\phi\rangle\right|^2 + \left|\langle\phi|\left\{\hat{A},\hat{B}\right\}|\phi\rangle\right|^2 = 4\left|\langle\phi|\hat{A}\hat{B}|\phi\rangle\right|^2$$

By Cauchy-Schwarz inequality

$$\left|\langle\phi|\left[\hat{A},\hat{B}\right]|\phi\rangle\right|^2+\left|\langle\phi|\left\{\hat{A},\hat{B}\right\}|\phi\rangle\right|^2=\left|\langle\phi|\hat{A}\hat{B}|\phi\rangle\right|^2\leq\langle\phi|\hat{A}^2|\phi\rangle\langle\phi|\hat{B}^2|\phi\rangle$$

and using the previous relation and dropping the term $\left|\langle\phi|\left\{\hat{A},\hat{B}\right\}|\phi\rangle\right|^2$ we get

$$\left|\langle\phi|\left[\hat{A},\hat{B}\right]|\phi\rangle\right|^2\leq 4\langle\phi|\hat{A}^2|\phi\rangle\langle\phi|\hat{B}^2|\phi\rangle$$

Suppose $\hat{C}$ and $\hat{D}$ are two observables. Substituting $\hat{A}=\hat{C}-\langle\hat{C}\rangle$ and $\hat{B}=\hat{D}-\langle\hat{D}\rangle$, where $\langle\hat{C}\rangle=\langle\phi|\hat{C}|\phi\rangle$ and $\langle\hat{D}\rangle=\langle\phi|\hat{D}|\phi\rangle$, we get

$$\begin{aligned}\langle\phi|\hat{A}^2|\phi\rangle &= \langle\phi|\left(\hat{C}^2-2\hat{C}\langle\hat{C}\rangle+\langle\hat{C}\rangle^2\right)|\phi\rangle=\langle\phi|\hat{C}^2|\phi\rangle-\langle\phi|\hat{C}|\phi\rangle^2=\langle\hat{C}^2\rangle-\langle\hat{C}\rangle^2,\\ \langle\phi|\hat{B}^2|\phi\rangle &= \langle\phi|\left(\hat{D}^2-2\hat{D}\langle\hat{D}\rangle+\langle\hat{D}\rangle^2\right)|\phi\rangle=\langle\phi|\hat{D}^2|\phi\rangle-\langle\phi|\hat{D}|\phi\rangle^2=\langle\hat{D}^2\rangle-\langle\hat{D}\rangle^2.\end{aligned}$$

Inserting these into the equation

$$\left|\langle\phi|\left[\hat{A},\hat{B}\right]|\phi\rangle\right|^2 \leq 4\langle\phi|\hat{A}^2|\phi\rangle\langle\phi|\hat{B}^2|\phi\rangle$$

and taking the square root of both sides of the inequality, we obtain the **Heisenberg uncertainty relation**

$$\Delta(\hat{C})\Delta(\hat{D}) \geq \frac{\left|\langle\phi|\left[\hat{C},\hat{D}\right]|\phi\rangle\right|}{2}$$

where $\Delta(\hat{C}) = \sqrt{\langle\hat{C}^2\rangle - \langle\hat{C}\rangle^2} = \sqrt{\langle\phi|\hat{C}^2|\phi\rangle - \langle\phi|\hat{C}|\phi\rangle^2}$ and $\Delta(\hat{D}) = \sqrt{\langle\phi|\hat{D}^2|\phi\rangle - \langle\phi|\hat{D}|\phi\rangle^2}$.

Example:

(i) Consider the observables $\hat{X} = \sigma_x$ and $\hat{Y} = \sigma_y$ when measured for the qubit state $|0\rangle$.

We know, or we can easily calculate, that $\left[\hat{X}, \hat{Y}\right] = 2i\hat{Z}$, where $\hat{Z} = \sigma_z$, so the uncertainty relation is

$$\Delta(\hat{X})\Delta(\hat{Y}) \geq \frac{\left|\langle 0| \left[\hat{X}, \hat{Y}\right] |0\rangle\right|}{2} = \langle 0|\hat{Z}|0\rangle = 1$$

(ii) Consider the observables $\hat{x}$ and $\hat{p}$in a state $|\psi\rangle$

$$\Delta(\hat{x})\Delta(\hat{p}) \geq \frac{\left|\langle\psi| \left[\hat{x}, \hat{p}\right] |\psi\rangle\right|}{2} = \frac{\hbar}{2}$$

Positive Operator-Valued Measure (POVM) measurements

Suppose a measurement described by the set of measurement operators $\{\hat{M}_m\}$ is performed upon a quantum system in the state $|\phi\rangle$.

Then the probability that result m occurs is $p_m = \langle\phi|\hat{M}_m^\dagger \hat{M}_m|\phi\rangle$.

Let us define

$$\hat{E}_m = \hat{M}_m^\dagger \hat{M}_m$$

then E_m is a positive operator such that

$$\sum_m \hat{E}_m = 1 \quad \text{and} \quad p_m = \langle\phi|\hat{E}_m|\phi\rangle$$

The set $\{E_m\}$ is known as a Positive Operator-Valued Measure or POVM.

The set of operators $\hat{E}_m$ which are known as POVM elements associated with the measurement, are sufficient to determine the probabilities of different measurement outcomes.

Example: Projective measurement

$$\hat{E}_m = \hat{P}_m$$

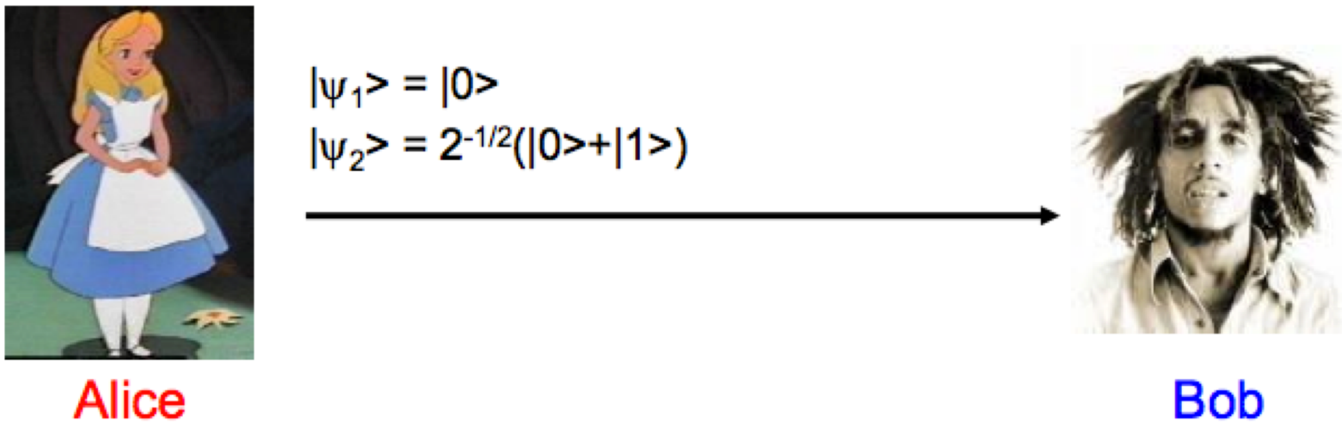
POVM measurement: example

Alice sends one of the states below

$$
\begin{aligned}
|\psi_1\rangle &= |0\rangle \\
|\psi_2\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)
\end{aligned}
$$

to Bob who however can not distinguish them reliably as they are not orthogonal.

However, he can perform a measurement that distinguishes the states some of the time, and never makes an error of mis-identification.



Consider the POVM

$$
\begin{aligned}
\hat{E}_1 &= \frac{\sqrt{2}}{1+\sqrt{2}}|1\rangle\langle 1| \\
\hat{E}_2 &= \frac{\sqrt{2}}{\left(1+\sqrt{2}\right)}\frac{|0\rangle - |1\rangle}{\sqrt{2}}\frac{\langle 0| - \langle 1|}{\sqrt{2}} \\
\hat{E}_3 &= \hat{I} - \hat{E}_1 - \hat{E}_2
\end{aligned}
$$

If the result of the measurement is E_1, then the state was $|\psi_2\rangle$, and if the result E_2 occurs then the state was $|\psi_1\rangle$. Some of the time however, Bob will obtain the result E_3 from which he can infer nothing about the state.

Measurement and quantum circuit

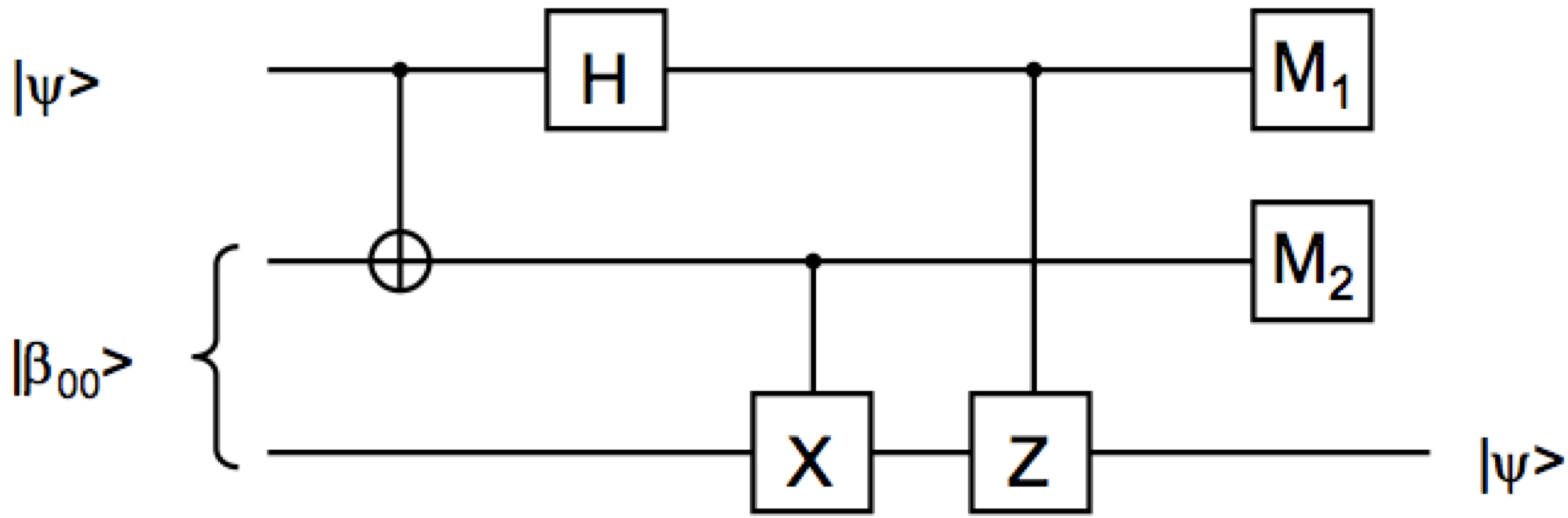
Principle of deferred measurement

Measurement can always be moved from an intermediate stage of a quantum circuit to the end of the circuit; if the measurement results are used at any stage of the circuit then the classically controlled operations can be replaced by conditional quantum operations.

Principle of implicit measurement

Without loss of generality, any unterminated quantum wires, that is, qubits that are not measured, at the end of a quantum circuit may be assumed to be measured.

Example: Principle of deferred measurement in quantum teleportation circuit



Measurement in other then computational basis

Recipe:
first unitarily transform from the basis you wish to perform a measurement in to the computational basis, and then measure qubits in the computational basis.

Example: Measurement in the Bell basis in the superdense coding protocol