QUANTUM MECHANICS FOUNDATIONS OF QUANTUM INFORMATION PROCESSING

STATES

FIRST POSTULATE

At a fixed time t, the state of a physical system is defined by specifying a ket $|\psi(t)\rangle$ belonging to the state space $\mathcal{H}$.

The state space is a space of all possible states of a given physical system, and it is a Hilbert space, i.e.

(1) a vector space over the field of complex numbers $\mathbb{C}$
(2) with inner product, and
(3) with a norm and a metric induced by the inner product, and
(4) it is also a complete space (relevant to infinite dimensions).

Definition of a vector space.

A vector space over the field of complex numbers $\mathbb{C}$ is a set of elements, called vectors, with an operation of *addition*, which for each pair of vectors $|\psi\rangle$ and $|\phi\rangle$ specifies a vector $|\psi\rangle + |\phi\rangle$, and an operation of *scalar multiplication*, which for each vector $|\psi\rangle$ and a number $c \in \mathbb{C}$ specifies a vector $c|\psi\rangle$ such that (s.t.)

1) $|\psi\rangle + |\phi\rangle = |\phi\rangle + |\psi\rangle$
2) $|\psi\rangle + (|\phi\rangle + |\chi\rangle) = (|\psi\rangle + |\phi\rangle) + |\chi\rangle$
3) there is a unique zero vector s.t. $|\psi\rangle + 0 = |\psi\rangle$
4) $c(|\psi\rangle + |\phi\rangle) = c|\psi\rangle + c|\phi\rangle$
5) $(c + d)|\psi\rangle = c|\psi\rangle + d|\psi\rangle$
6) $c(d|\psi\rangle) = (cd)|\psi\rangle$
7) $1.|\psi\rangle = |\psi\rangle$
8) $0.|\psi\rangle = 0$

Example:

A set of N-tuples of complex numbers.

An inner product.

Dirac bra-ket notation:

$$\begin{aligned} |\psi\rangle, |\phi\rangle &\in \mathcal{H} \\ \langle\phi|\psi\rangle &\in \mathbb{C} \end{aligned}$$

A bra $\langle\phi|$ is the adjoint of a ket $|\phi\rangle$, e.g.

$$\begin{aligned} \text{if} \qquad & |\psi\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle, \\ \text{then} \qquad & \langle\psi| = c_1^*\langle\phi_1| + c_2^*\langle\phi_2| \end{aligned}$$

We call $|\phi_1\rangle$ and $|\phi_2\rangle$ a **basis** (or basis elements) of $\mathcal{H}$ if and only if

$$\begin{aligned} \text{span}\{|\phi_1\rangle, |\phi_2\rangle\} &= \mathcal{H} \\ \text{and } \langle\phi_i|\phi_j\rangle &= \delta_{ij} \end{aligned}$$

where δ_{ij} is the Kronecker delta-symbol. And with a norm and metric induced by the inner product.

Norm:

$$\begin{aligned} \text{e.g. } \langle\phi_i|\phi_j\rangle &= \delta_{ij} \text{ i.e.} \\ \langle\phi_1|\phi_1\rangle^{1/2} &= \|\phi_1\| = 1 \\ &\equiv \text{the norm of } |\phi_1\rangle \end{aligned}$$

If the norm is 1, the state is said to be normalized, i.e. its length equals 1.

Two vectors are orthogonal if their inner product is zero. A set of mutually orthogonal vectors of unit norm is said to be orthonormal.

Metric: a metric is a map which assigns to each pair of vectors $|\psi\rangle$, $|\phi\rangle$ a scalar $\rho \geq 0$ such that

1. $\rho\left(|\psi\rangle, |\phi\rangle\right) = 0$ iff $|\psi\rangle = |\phi\rangle$;

2. $\rho\left(|\psi\rangle, |\phi\rangle\right) = \rho\left(|\phi\rangle, |\psi\rangle\right)$

3. $\rho\left(|\psi\rangle, |\chi\rangle\right) \leq \rho\left(|\psi\rangle, |\phi\rangle\right) + \rho\left(|\phi\rangle, |\chi\rangle\right)$ (triangle identity)

We say that the metric is induced by the norm if

$$\rho\left(|\psi\rangle, |\phi\rangle\right) = \left\| |\psi\rangle - |\phi\rangle \right\|$$

So the Hilbert space is normed and a metric space. What else?

It is also a complete space so every Cauchy sequence of vectors, i.e.

$$|||\psi_n\rangle - |\psi_m\rangle|| \to 0 \quad \text{as} \quad m, n \to \infty$$

converges to a limit vector in the space.
(We need this condition to be able to handle systems with infinite-dimensional Hilbert spaces, i.e. with infinite degrees of freedom.)

Can we be more concrete about quantum states? What really is a ket $|\psi\rangle$?

Now, we need the concept of representation.
Let us say we have the Hilbert space $\mathcal{H}$ and the basis

$$\mathcal{B} \quad = \quad \{|\phi_1\rangle, |\phi_2\rangle\}$$

and we have a ket

$$|\psi\rangle \in \mathcal{H}$$

which we wish to express in the representation given by the basis $\mathcal{B}$.
We use the completeness relation

$$\sum_i |\phi_i\rangle\langle\phi_i| = \hat{1}$$

as follows

$$\begin{aligned} |\psi\rangle &= \sum_i |\phi_i\rangle \underbrace{\langle\phi_i|\psi\rangle}_{\text{a number}\in\mathbb{C}} \\ &= \sum_i c_i|\phi_i\rangle \end{aligned}$$

Our state becomes a specific superposition of the basis set elements, i.e. we have expanded $|\psi\rangle$ in terms of $\{|\phi_i\rangle\}$.

Quantum bit

Quantum bit or **qubit** is a two dimensional Hilbert space $\mathcal{H}^2 \simeq \mathbb{C}^2$. Its values are vectors, states, or kets, from this Hilbert space:

$$|\phi\rangle = c_0|0\rangle + c_1|1\rangle = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$

The vectors $|0\rangle$ and $|1\rangle$ are the basis vectors from the **standard computational basis**:

$$\mathcal{B} = \{|0\rangle, |1\rangle\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

so that $\mathcal{H}^2 = \text{span}\,(\mathcal{B})$. The conjugate bra $\langle\phi| = c_0^*\langle 0| + c_1^*\langle 1| = \begin{pmatrix} c_0^* & c_1^* \end{pmatrix}$
The coefficients c_0 and c_1 are complex numbers, $c_0, c_1 \in \mathbb{C}$, satisfying $|c_0|^2 + |c_1|^2 = 1$.

Recall that multiplying a quantum states by global phase, a complex number of unit modulus $e^{i\theta}$, has no observable consequences:

$$|\phi\rangle = c_0|0\rangle + c_1|1\rangle \quad \to \quad |\phi'\rangle = e^{i\theta}|\phi\rangle = c_0 e^{i\theta}|0\rangle + c_1 e^{i\theta}|1\rangle = c_0'|0\rangle + c_1'|1\rangle$$

The probability of obtaining the measurement result 0 or 1 when measuring the qubit in the standard basis remains the same:

$$\begin{aligned} \left|c_0'\right|^2 &= c_0'^* \, c_0' = c_0^* e^{-i\theta} \, c_0 e^{i\theta} = c_0^* c_0 = \left|c_0\right|^2 \\ \left|c_1'\right|^2 &= c_1'^* \, c_1' = c_1^* e^{-i\theta} \, c_1 e^{i\theta} = c_1^* c_1 = |c_1|^2 \end{aligned}$$

The expectation value or average value of an observable $\hat{O}$, obtained from its repeated measurement on the qubits in an equally prepared state, is also invariant with the global phase:

$$< \hat{O} >_{\phi'} = \langle \phi' | \hat{O} | \phi' \rangle = e^{-i\theta} e^{i\theta} \langle \phi | \hat{O} | \phi \rangle =< \hat{O} >_{\phi}$$

This suggests that we need three real numbers to specify a state of one qubit.

Density operator/matrix

We can represent a qubit state $|\phi\rangle$, and any quantum state, by the projector onto the one-dimensional subspace it spans:

$$\begin{aligned}\hat{\rho} &= |\phi\rangle\langle\phi| = (c_0|0\rangle + c_1|1\rangle)\left(c_0^*\langle 0| + c_1^*\langle 1|\right) \\ &= \left|c_0\right|^2 \; |0\rangle\langle 0| + c_0c_1^* \; |0\rangle\langle 1| + c_0^*c_1 \; |1\rangle\langle 0| + |c_1|^2 \; |1\rangle\langle 1|\end{aligned}$$

In matrix representation given by the standard computational basis, we have

$$\hat{\rho} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \begin{pmatrix} c_0^* & c_1^* \end{pmatrix} = \begin{pmatrix} \left|c_0\right|^2 & c_0c_1^* \\ & \\ c_0^*c_1 & |c_1|^2 \end{pmatrix}$$

We observe that the **norm** of a state is $\mathrm{Tr}(\hat{\rho}) = |c_0|^2 + |c_1|^2 = 1$ and also that $\rho_{10} = \rho_{01}^*$.

Bloch representation

The single-qubit density matrix can be decomposed as follows

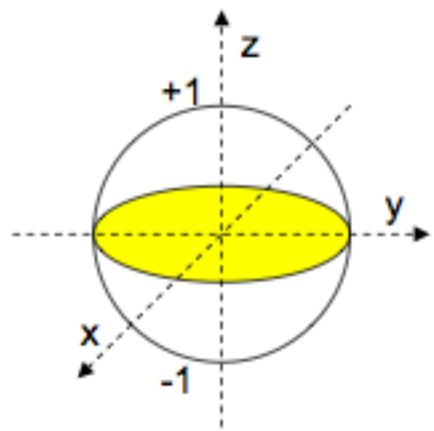
$$\hat{\rho} = \frac{1}{2}\left(\hat{I} + \vec{r}\,.\,\vec{\sigma}\right) = \frac{1}{2}\left(\hat{I} + r_x\,\sigma_x + r_y\,\sigma_y + r_z\,\sigma_z\right)$$

$$= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + r_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + r_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + r_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

where σ_x, σ_y, and σ_z are the Pauli matrices.

The vector $\vec{r} = \left(r_x, r_y, r_z\right)$ is called the **Bloch vector** and its components, real numbers between 0 between 1, are related to the density matrix elements as follows:

$$r_x = 2\,\mathrm{Re}\,(\rho_{10})$$
$$r_y = 2\,\mathrm{Im}\,(\rho_{10})$$
$$r_z = \rho_{00} - \rho_{11}$$



Examples

To construct their Bloch representation of pure states of one qubit (mixed states will come later) we use

$$
\begin{aligned}
r_x &= 2\,\mathrm{Re}\,(\rho_{10}) = 2\,\mathrm{Re}\left(c_0^* c_1\right) \\
r_y &= 2\,\mathrm{Im}\,(\rho_{10}) = 2\,\mathrm{Im}\left(c_0^* c_1\right) \\
r_z &= \rho_{00} - \rho_{11} = \left|c_0\right|^2 - |c_1|^2
\end{aligned}
$$

1. $|\phi\rangle = |0\rangle$

$$\hat{\rho} = |0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1^* & 0^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \vec{r} = (0, 0, 1)$$

2. $|\phi\rangle = |1\rangle$

$$\hat{\rho} = |1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0^* & 1^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \vec{r} = (0, 0, -1)$$

3. $|\phi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$

$$\hat{\rho} = |\phi\rangle\langle\phi| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}^* & \frac{1}{\sqrt{2}}^* \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ \\ 1 & 1 \end{pmatrix} \quad \Rightarrow \quad \vec{r} = (1, 0, 0)$$

4. $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$$\hat{\rho} = |\phi\rangle\langle\phi| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}^* & -\frac{1}{\sqrt{2}}^* \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \Rightarrow \quad \vec{r} = (-1, 0, 0)$$

5. $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

$$\hat{\rho} = |\phi\rangle\langle\phi| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ i\,\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}^* & \left(i\,\frac{1}{\sqrt{2}}\right)^* \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \quad \Rightarrow \quad \vec{r} = (0, 1, 0)$$

Bloch sphere

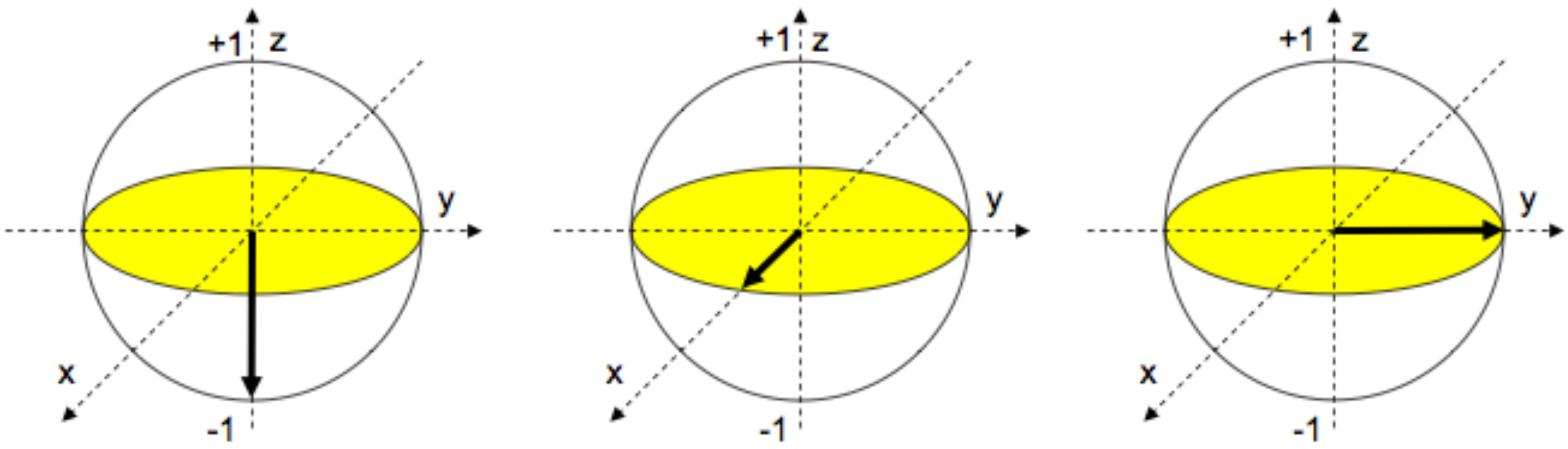
The set of all Bloch vectors for single qubit pure states form a surface of a sphere of unit radius.

<u>Examples</u>

$|\phi\rangle = |1\rangle$ $\quad \vec{r} = (0, 0, -1)$

$|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ $\quad \vec{r} = (1, 0, 0)$

$|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ $\quad \vec{r} = (0, 1, 0)$



Composition of Hilbert spaces

A tensor product of vector space $\mathcal{V}$ and $\mathcal{U}$ is a vector space $\mathcal{W}$ whose dimension is (dim$\mathcal{V}$).(dim$\mathcal{U}$).

Let $\mathcal{B}_{\mathcal{U}} = \{|u_1\rangle, |u_2\rangle, \ldots, |u_n\rangle\}$ be a basis of $\mathcal{U}$ and $\mathcal{B}_{\mathcal{V}} = \{|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle\}$ be a basis of $\mathcal{V}$, then a basis of $\mathcal{W} = \mathcal{U} \otimes \mathcal{V}$ is $\mathcal{B}_{\mathcal{W}} = \{|u_1 v_1\rangle, |u_1 v_2\rangle, \ldots, |u_n v_n\rangle\}$ where $|u_k v_l\rangle = |u_k\rangle \otimes |v_l\rangle$.

Example

Let $\mathcal{B}_{\mathcal{U}} = \{|0\rangle, |1\rangle\}$, $\mathcal{B}_{\mathcal{V}} = \{|0\rangle, |1\rangle\}$, then $\mathcal{B}_{\mathcal{W}} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

Qubits

A quantum state of n qubits is a vector in 2^n-dimensional Hilbert space:

$$\bigotimes_{k=1}^{n} \mathcal{H}^2 = \mathcal{H}^2 \otimes \mathcal{H}^2 \otimes ...\mathcal{H}^2 \quad \text{(n-times)} = \mathcal{H}^{2^n}$$

The **standard computational basis** of n-qubit Hilbert space:

$$\mathcal{B}_{\mathcal{H}^{2^n}} = \{|0\ldots000\rangle, |00\ldots001\rangle, |00\ldots010\rangle, |00\ldots011\rangle, \ldots |1\ldots111\rangle\}$$

<u>Example:</u> The standard basis of a two-qubit Hilbert space

$$\mathcal{B}_{\mathcal{H}^4} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

Examples of two-qubit states:

(i) Composite product states

$$
\begin{aligned}
|\phi\rangle &= |0\rangle \otimes |0\rangle = |0\rangle|0\rangle = |00\rangle \\
|\psi\rangle &= (c_0|0\rangle + c_1|1\rangle) \otimes |0\rangle = c_{00}|00\rangle + c_{10}|10\rangle
\end{aligned}
$$

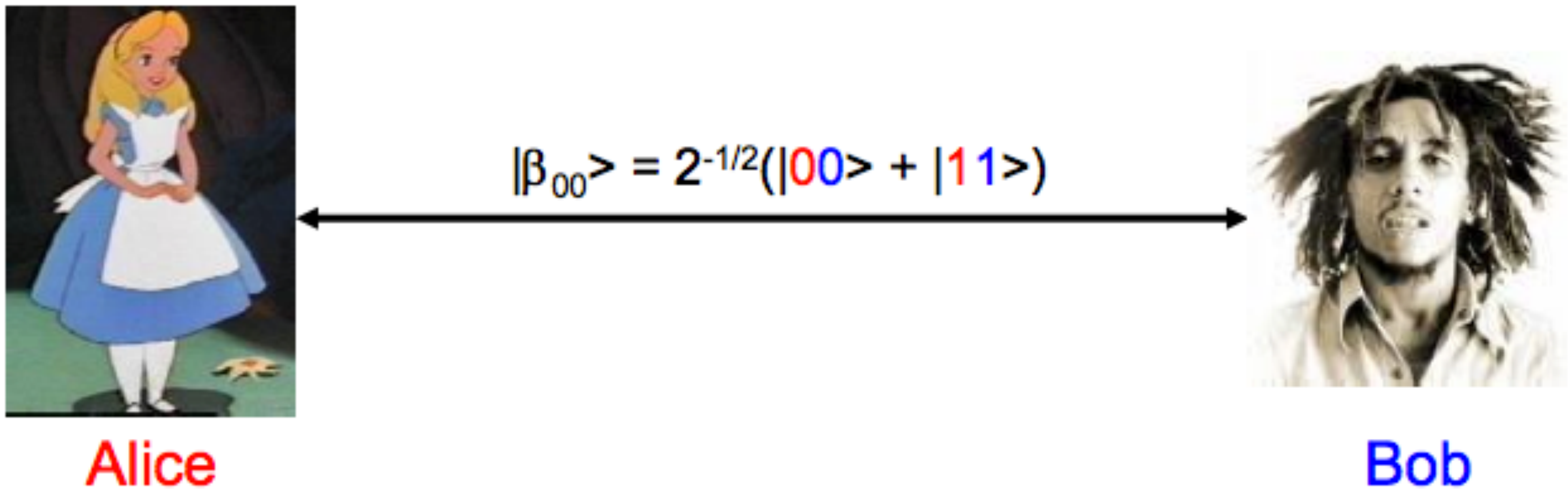
(ii) Entangled states: the Bell states

$$
\begin{aligned}
|\beta_{00}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
|\beta_{01}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
|\beta_{10}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
|\beta_{11}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)
\end{aligned}
$$

Superdense coding

Task:
Alice wants to send two classical bits of information, that is one of the bit strings $\{00, 01, 10, 11\}$, to Bob.

Resources:
i) Alice and Bob share two qubits in the Bell state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
ii) Alice can send her one quantum bit to Bob.



Superdense coding protocol

1. Alice and Bob share two qubits in the Bell state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

2. Depending on what bit string, $00, 01, 10, 11$, Alice wants to send to Bob, she applies one of the following transformations to her qubit:

$$\begin{array}{llllll} 00: & \hat{I}: & |\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) & \rightarrow & |\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ 01: & \hat{Z}: & |\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) & \rightarrow & |\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\ 10: & \hat{X}: & |\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) & \rightarrow & |\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ 11: & \hat{Z}\hat{X} = i\hat{Y}: & |\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) & \rightarrow & |\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{array}$$

3. The resulting Bell states are orthogonal and hence Bob can distinguish them by a measurement in the Bell basis.

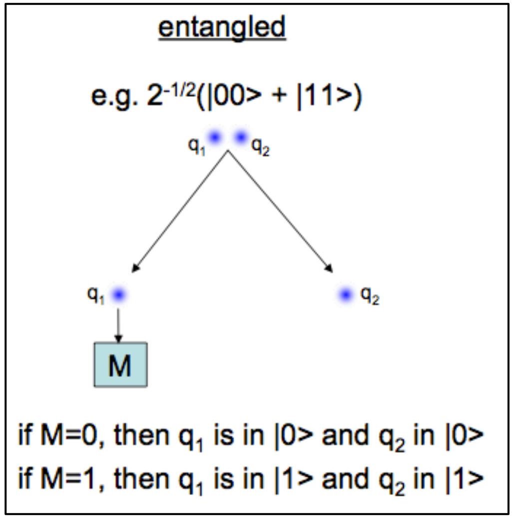
One qubit is sufficient to transmit two bits of classical information.

Einstein-Podolsky-Rosen paradox

The measurement of one qubit of an entangled two-qubit state completely determines the state of the other qubit after the measurement even if both qubits are spatially separated. This implies that the first qubit communicates with the other instantaneously, that is, faster than light, across the space ⇒ **spooky action at a distance** - A. Einstein.

Resolving the paradox:
Hidden variables theory: Quantum mechanics can not be complete. There must be some unknown mechanism acting on quantum mechanical variables to give rise to observable effects of noncommutative quantum observables like Heisenberg uncertainty principle.

Bell inequalities: No hidden variable theory.

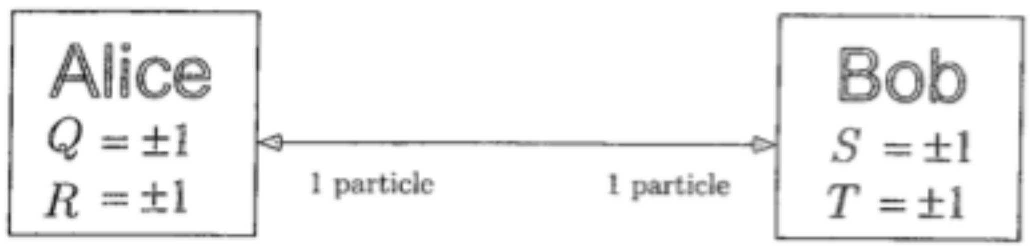


Bell inequalities John S. Bell 1962

Alice and Bob share a two-particle system.

Each can perform one of two different measurements and they can decide which measurement to perform by flipping a coin once they receive a particle. The measurement outcome can be $+1$ or -1.

Alice can measure physical properties of her particle P_Q or P_R, and Bob can measure properties P_S or P_T of his particle; both measurements take place at the same time.



We calculate the quantity

$$QS + RS + RT - QT = (Q + R)S + (R - Q)T$$

Because $R, Q = \pm 1$ it follows that either

$$(Q + R)S = 0 \quad \text{or} \quad (R - Q)T = 0$$

In either case

$$QS + RS + RT - QT = \pm 2$$

Suppose that $p(q, r, s, t)$ is the probability that before the measurements, the system is in the state where $Q = q$, $R = r$, $S = s$, and $T = t$.
The mean value

$$E(QS + RS + RT - QT) = \sum_{q,r,s,t} p(q, r, s, t)(qs + rs + rt - qt) \leq \sum_{q,r,s,t} p(q, r, s, t) \times 2 = 2$$

Also

$$\begin{aligned} E(QS + RS + RT - QT) &= \sum_{q,r,s,t} p(q, r, s, t)qs + \sum_{q,r,s,t} p(q, r, s, t)rs \\ &\quad + \sum_{q,r,s,t} p(q, r, s, t)rt - \sum_{q,r,s,t} p(q, r, s, t)qt \\ &= E(QS) + E(RS) + E(RT) - E(QT) \end{aligned}$$

Comparing both gives the **Bell inequality**

$$E(QS) + E(RS) + E(RT) - E(QT) \leq 2$$

Now let Alice and Bob share a quantum system of two qubits in the state

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \qquad (\ = \ |\beta_{11}\rangle \ )$$

They perform measurements of the following observables:

Alice: $Q = \hat{Z}_1 \quad R = \hat{X}_1$ Bob: $S = \frac{-\hat{Z}_2 - \hat{X}_2}{\sqrt{2}} \quad T = \frac{\hat{Z}_2 - \hat{X}_2}{\sqrt{2}}$

The expectation values of these observables are

$$< QS >= \frac{1}{\sqrt{2}}; \quad < RS >= \frac{1}{\sqrt{2}}; \quad < RT >= \frac{1}{\sqrt{2}}; \quad < QT >= -\frac{1}{\sqrt{2}}$$

Thus **quantum mechanical systems violate the Bell inequality**:

$$< QS > + < RS > +RT > - < QT >= 2\sqrt{2}$$

Experiment: Alain Aspect 1982.

Evaluation of the expectation values:

$$
\begin{aligned}
< QS > &= \langle\psi|\frac{-\hat{Z}_1 \otimes \hat{Z}_2 - \hat{Z}_1 \otimes \hat{X}_2}{\sqrt{2}}|\psi\rangle = \frac{1}{2\sqrt{2}}(\langle 01| - \langle 10|)(-\hat{Z}_1 \otimes \hat{Z}_2 - \hat{Z}_1 \otimes \hat{X}_2)((|01\rangle - |10\rangle)) \\
&= \frac{1}{2\sqrt{2}}(\langle 01|(-\hat{Z}_1 \otimes \hat{Z}_2)|01\rangle + \langle 10|(-\hat{Z}_1 \otimes \hat{Z}_2)|10\rangle) = \frac{1}{2\sqrt{2}}(1+1) = \frac{1}{\sqrt{2}} \\
< RS > &= \langle\psi|\frac{-\hat{X}_1 \otimes \hat{Z}_2 - \hat{X}_1 \otimes \hat{X}_2}{\sqrt{2}}|\psi\rangle = \frac{1}{\sqrt{2}} \\
< RT > &= \langle\psi|\frac{\hat{X}_1 \otimes \hat{Z}_2 - \hat{X}_1 \otimes \hat{X}_2}{\sqrt{2}}|\psi\rangle = \frac{1}{\sqrt{2}} \\
< QT > &= \langle\psi|\frac{\hat{Z}_1 \otimes \hat{Z}_2 - \hat{Z}_1 \otimes \hat{X}_2}{\sqrt{2}}|\psi\rangle = -\frac{1}{\sqrt{2}}
\end{aligned}
$$

Entanglement on bipartite systems

Theorem: Schmidt's decomposition

Suppose $|\psi\rangle$ is a pure state of a bipartite system, AB. Then there exist orthonormal states $|i_A\rangle$ for system A, and $|i_B\rangle$ for system B such that

$$|\psi\rangle = \sum_i \lambda_i \, |i_A\rangle|i_B\rangle$$

where λ_i are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$ known as the **Schmidt coefficients**.

The number of non-zero values λ_i is called the **Schmidt number**.

Proof

Let us assume for the sake of simplicity that the Hilbert spaces for the system A and the system B have the same dimension. Let $\{|j\rangle\}$ and $\{|k\rangle\}$ be any fixed basis for systems A and B, respectively. Then $|\psi\rangle$ can be written as

$$|\psi\rangle = \sum_{jk} a_{jk} \, |j\rangle|k\rangle$$

for some matrix a of complex numbers a_{jk}.

By singular value decomposition, $a = udv$, where d is a diagonal matrix with non-negative real elements, and u and v are unitary matrices. Thus

$$|\psi\rangle = \sum_{ijk} u_{ji} d_{ii} v_{ik} \, |j\rangle|k\rangle$$

Defining $|i_A\rangle = \sum_j u_{ji}\,|j\rangle$ and $|i_B\rangle = \sum_k v_{ik}\,|k\rangle$, and $\lambda_i = d_{ii}$ we get

$$|\psi\rangle = \sum_i \lambda_i\,|i_A\rangle|i_B\rangle$$

Both $|i_A\rangle$ and $|i_B\rangle$ form orthonormal sets. This follows from the unitarity of u and v and orthonormality of $|j\rangle$ and $|k\rangle$. Q.E.D.

If the Schmidt number is 1, then the quantum state of the bipartite system is a product state, otherwise it is an entangled states.

Examples

(i) Schmidt number = 1

a) Let us have the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle$. The matrix a is then

$$a = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

Now we construct the matrix $aa^\dagger$

$$aa^\dagger = udvv^\dagger d^\dagger u^\dagger = ud^2u^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

where $d = d^\dagger$ because d is diagonal matrix with real entries.The matrix $aa^\dagger$ is already diagonal and has one nonzero eigenvalue.Thus the state $|\psi\rangle$ is a **product state**.

b) Let us have the state $|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)$. The matrix a is then

$$a = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Now we construct the matrix $aa^\dagger$

$$aa^\dagger = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

The matrix $aa^\dagger$ is not diagonal so we have to diagonalize it. The eigenvalues are given as the roots of the characteristic equation

$$\det\left(aa^\dagger - \lambda\hat{I}\right) = \left(\frac{1}{2} - \lambda\right)^2 - \frac{1}{4} = 0$$

we get $\lambda_1 = 1$ and $\lambda_2 = 0$, so the state is again a **product state**.

(ii) Schmidt number = 2

a) Let us have the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle$. The matrix a is then

$$a = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Now we construct the matrix $aa^\dagger$

$$aa^\dagger = udvv^\dagger d^\dagger u^\dagger = ud^2u^\dagger = \begin{pmatrix} \frac{1}{2} & 0 \\ \\ 0 & \frac{1}{2} \end{pmatrix}$$

The matrix $aa^\dagger$ is diagonal and has **two** nonzero eigenvalues.Thus the state $|\psi\rangle$ is **entangled**.

b) Let us have the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - i|10\rangle$. The matrix a and $a^\dagger$ are then

$$a = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -i\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \qquad a^\dagger = \begin{pmatrix} 0 & i\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Now we construct the matrix $aa^\dagger$

$$aa^\dagger = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

The Schmidt number thus equals to 2, and therefore the state $|\psi\rangle$ is **entangled**.