OPEN QUANTUM SYSTEMS
QUANTUM OPERATIONS

Quantum state tomography of a qubit

Experimental determination of an unknown qubit state ρ:

- using a single copy of ρ, it is impossible to characterize the state;

- using many qubits in an equally prepared state, it is possible to estimate ρ using

quantum state tomography.

The set

$$\frac{I}{\sqrt{2}}, \frac{X}{\sqrt{2}}, \frac{Y}{\sqrt{2}}, \frac{Z}{\sqrt{2}}$$

is an orthonormal set of operators with respect to the Hilbert-Schmidt inner product $(A, B) = \mathrm{tr}\,(A^\dagger B)$. It can be therefore used to expand the density matrix as

$$\rho = \frac{1}{2}\left[\mathrm{tr}\,(\rho)I + \mathrm{tr}\,(X\rho)X + \mathrm{tr}\,(Y\rho)Y + \mathrm{tr}\,(Z\rho)Z\right]$$

where the quantities $\mathrm{tr}\,(X\rho)$, $\mathrm{tr}\,(Y\rho)$, and $\mathrm{tr}\,(Z\rho)$ have the interpretation of the average value of the observables X, Y and Z respectively. To get estimates of these quantities, the measurements of X, Y and Z need to be performed repeatedly on a large number m of equally prepared states ρ. The uncertainty of the result is decreasing as $1/\sqrt{m}$ via the central limit theorem.

The density matrix can be reconstructed from the measurement results.

Quantum process tomography

Experimental identification of the dynamics of quantum systems, that is, a set of operation elements $\{E_i\}$ for $\mathcal{E}$.

In general for d dimensional quantum system $\mathcal{H}^d$

- we choose d^2 pure quantum states $\{|\psi_j\rangle\}$, chosen so that the corresponding density matrices $\{|\psi_j\rangle\langle\psi_j|\}$ form a basis for the space of matrices;
- then we subject each of the states to the process we wish to characterize;
- after completion of this process, we run quantum state tomography to determine the output state $\mathcal{E}(|\psi_j\rangle\langle\psi_j|)$ from the process.

A way of determining a useful representation of $\mathcal{E}$ – χ-matrix representation:

$$\mathcal{E}(\rho) = \sum_i E_i \, \rho \, E_i^\dagger$$

To determine the E_i from measurable parameters, we can consider using a fixed set of $\tilde{E}_i$, which form a basis for the set of operators on the Hilbert space:

$$E_i = \sum_m e_{im} \, \tilde{E}_m$$

where e_{im} are complex numbers. The quantum operation is then given as

$$\mathcal{E}(\rho) = \sum_{mn} \tilde{E}_m \, \rho \, \tilde{E}_n^\dagger \, \chi_{mn} \quad \text{where} \quad \chi_{mn} = \sum_i e_{im} \, e_{in}^*$$

The matrix χ completely describes $\mathcal{E}(\rho)$ once the set of operators $\tilde{E}_i$ has been fixed.

In general, χ will contain $d^4 - d^2$ independent real parameters:

- a general linear map between $d \times d$ matrices is described by d^4 parameters,

- the fact that ρ remains self-adjoint with $\mathrm{tr}\,\rho = 1$ gives additional d^2 constraints, that is, the completeness relation

$$\sum_i E_i^\dagger E_i = I$$

is satisfied giving d^2 constraints.

We will see

(i) how to determine χ experimentally,

(ii) how an operator sum representation can be recovered once the χ matrix is known.

Let $\rho_j,\ j = 1, ..., d^2$, be fixed linearly independent basis for the space of $d \times d$ matrices.

A convenient choice of operators is $|n\rangle\langle m|$.

Experimentally, the output state $\mathcal{E}(|n\rangle\langle m|)$ may be determined by preparing the input states

$$\begin{aligned}
&|n\rangle \\
&|m\rangle \\
&|+\rangle = \frac{1}{\sqrt{2}}(|n\rangle + |m\rangle) \\
&|-\rangle = \frac{1}{\sqrt{2}}(|n\rangle + i|m\rangle)
\end{aligned}$$

and forming linear combinations of $\mathcal{E}(|n\rangle\langle n|)$, $\mathcal{E}(|m\rangle\langle m|)$, $\mathcal{E}(|+\rangle\langle +|)$, $\mathcal{E}(|-\rangle\langle -|)$:

$$\mathcal{E}(|n\rangle\langle m|) = \mathcal{E}(|+\rangle\langle +|) + i\,\mathcal{E}(|-\rangle\langle -|) - \frac{1+i}{2}\mathcal{E}(|n\rangle\langle n|) - \frac{1+i}{2}\mathcal{E}(|m\rangle\langle m|)$$

Thus it is possible to determine $\mathcal{E}(\rho_j)$ by state tomography for each ρ_j.

Furthermore, each $\mathcal{E}(\rho_j)$ may be expressed as a linear combination of the basis states

$$\mathcal{E}(\rho_j) = \sum_k \lambda_{jk}\, \rho_k$$

and since $\mathcal{E}(\rho_j)$ is known from the state tomography, λ_{jk} can be determined by linear algebra. We may write

$$\tilde{E}_m\, \rho_j\, \tilde{E}_n^\dagger = \sum_k \beta_{jk}^{mn} \rho_k$$

where β_{jk}^{mn} are complex numbers which can be determined using linear algebra given the $\tilde{E}_m$ operators and the ρ_j operators. Combining the last expressions gives

$$\sum_k \sum_{mn} \chi_{mn}\, \beta_{jk}^{mn}\, \rho_k = \sum_k \lambda_{jk}\, \rho_k$$

From the linear independence of ρ_k, it follows that for each k

$$\sum_{mn} \beta_{jk}^{mn} \chi_{mn} = \lambda_{jk}$$

This relation is a necessary and sufficient condition for the matrix χ to give the correct quantum operation $\mathcal{E}$.

One may think of χ and λ as vectors, and β as a $d^4 \times d^4$ matrix with columns and rows indexed by mn and jk respectively.

χ can be obtained using κ which is a generalized inverse of the matrix β:

$$\chi_{mn} = \sum_{jk} \kappa_{jk}^{mn} \lambda_{jk}$$

Having determined the matrix χ, the operator-sum representation is obtained as follows. Let the unitary matrix $U^\dagger$ diagonalize χ

$$\chi_{mn} = \sum_{xy} U_{mx}\, d_x\, \delta_{xy}\, U^*_{ny}$$

From this it can be verified that

$$E_i = \sqrt{d_i} \sum_j U_{ji}\, \tilde{E}_j$$

are the operation elements for $\mathcal{E}$.

Quantum process tomography of a single qubit

We fix the following set of operators

$$\tilde{E}_0 = I \quad \tilde{E}_1 = X \quad \tilde{E}_2 = -iY \quad \tilde{E}_3 = Z$$

There are 12 parameters, given by χ, which determine an arbitrary single qubit quantum operation $\mathcal{E}$. These parameters may be measured by 4 sets of experiments.

For example, suppose the input states $|0\rangle$, $|1\rangle$, $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, and $|-\rangle = (|0\rangle + i\,|1\rangle)/\sqrt{2}$ are prepared , and the four matrices below are determined by tomography

$$\begin{aligned}
\rho_1' &= \mathcal{E}(|0\rangle\langle 0|) \\
\rho_4' &= \mathcal{E}(|1\rangle\langle 1|) \\
\rho_2' &= \mathcal{E}(|+\rangle\langle +|) - i\,\mathcal{E}(|-\rangle\langle -|) - \frac{1-i}{2}\left(\rho_1' + \rho_4'\right) \\
\rho_3' &= \mathcal{E}(|+\rangle\langle +|) + i\,\mathcal{E}(|-\rangle\langle -|) - \frac{1+i}{2}\left(\rho_1' + \rho_4'\right)
\end{aligned}$$

These correspond to $\rho'_j = \mathcal{E}(\rho_j)$, where

$$
\begin{aligned}
\rho_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
\rho_2 &= \rho_1 \, X \\
\rho_3 &= X \, \rho_1 \\
\rho_4 &= X \, \rho_1 \, X
\end{aligned}
$$

From the general equations we may determine β and similarly ρ'_j determines λ.

In our specific case, we can be more explicit.

Due to a particular choice of basis, and the Pauli representation of $\tilde{E}_i$, we may express the β matrix as

$$\beta = \Lambda \otimes \Lambda$$

where

$$\Lambda = \frac{1}{2}\begin{pmatrix} I & X \\ X & -I \end{pmatrix}$$

χ can then be conveniently expressed as

$$\chi = \Lambda \begin{pmatrix} \rho'_1 & \rho'_2 \\ \rho'_3 & \rho'_4 \end{pmatrix} \Lambda$$

in terms of block of matrices.

QUANTUM ERROR CORRECTION

Distance measures for quantum information

Static:	how different are two quantum states?
Dynamic:	how well has the information been preserved during dynamics?

Static measures

(i) Trace distance

The trace distance is a metric on the space of density operators

$$D(\rho_1, \rho_2) = \frac{1}{2} \operatorname{tr} |\rho_1 - \rho_2|$$

where $|A| = \sqrt{A^\dagger A}$.

Example: single qubit states in the Bloch representation

$$D(\rho_1, \rho_2) = \frac{1}{2}\,\mathrm{tr}\,|\rho_1 - \rho_2| = \frac{1}{4}\,\mathrm{tr}\,\left|(\vec{r}_1 - \vec{r}_2).\vec{\sigma}\right|$$

Since the term $(\vec{r}_1 - \vec{r}_2).\vec{\sigma}$ has the eigenvalues $\pm\left|\vec{r}_1 - \vec{r}_2\right|$, so the trace of its absolute value is then $2\left|\vec{r}_1 - \vec{r}_2\right|$, giving

$$D(\rho_1, \rho_2) = \frac{1}{2}\,\mathrm{tr}\,\left|\vec{r}_1 - \vec{r}_2\right|$$

Theorem: Trace preserving operations are contractive:

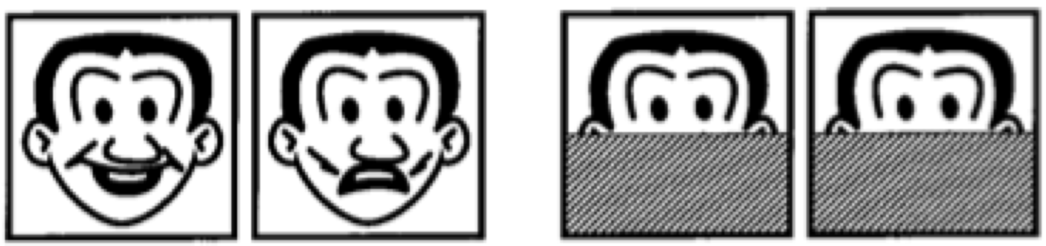
Suppose $\mathcal{E}$ is a trace preserving operation. Let ρ and σ be density operators. Then

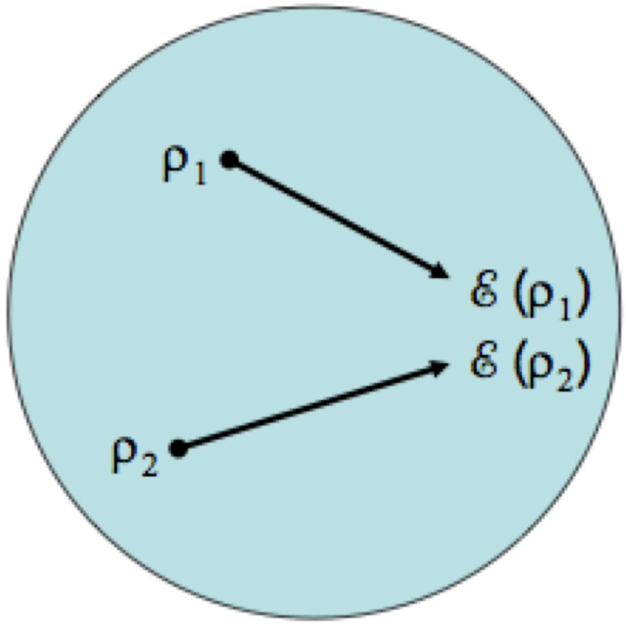
$$D\left(\mathcal{E}(\rho), \mathcal{E}(\sigma)\right) \leq D(\rho, \sigma)$$

that is, distinct states appear closer to each other if only a partial information about them is available.

Remark: Unitary invariance of the trace distance

$$D(U\rho U^{\dagger}, U\sigma U^{\dagger}) = D(\rho, \sigma)$$





Theorem: Strong convexity of the trace distance:

Let $\{p_i\}$ and $\{q_i\}$ be probability distributions over the same index set, and ρ_i and σ_i be density operators, with indices from the same set. Then

$$D\left(\sum_i p_i\rho_i, \sum_i q_i\sigma_i\right) \le D(p_i, q_i) + \sum_i p_i D(\rho_i, \sigma_i)$$

where $D(p_i, q_i)$ is the classical trace distance between the probability distributions $\{p_i\}$ and $\{q_i\}$.

(ii) Fidelity

$$F(\rho_1, \rho_2) = \mathrm{tr}\ \sqrt{\sqrt{\rho_1}\rho_2\ \sqrt{\rho_1}}$$

is not a metric on the space of density operators but it is still a good distance measure.

Example: Fidelity between a pure state $|\psi\rangle$ and a mixed state ρ

$$\begin{aligned} F\left(|\psi\rangle\langle\psi|, \rho\right) &= \mathrm{tr}\ \sqrt{\sqrt{|\psi\rangle\langle\psi|}\rho\ \sqrt{|\psi\rangle\langle\psi|}} = \mathrm{tr}\ \sqrt{|\psi\rangle\langle\psi|\rho|\psi\rangle\langle\psi|} = \mathrm{tr}\ \sqrt{\langle\psi|\rho|\psi\rangle|\psi\rangle\langle\psi|} \\ &= \sqrt{\langle\psi|\rho|\psi\rangle} \end{aligned}$$

The fidelity is the square roor of the expectation value of the density operator ρ with respect to the pure state $|\psi\rangle$.

Properties:

Unitary invariance:

$$F(U\rho_1 U^\dagger, U\rho_2 U^\dagger) = F(\rho_1, \rho_2)$$

Symmetry in the inputs:

$$F(\rho_2, \rho_1) = F(\rho_1, \rho_2)$$

Boundedness:

$$0 \leq F(\rho_1, \rho_2) \leq 1$$

where $F(\rho_1, \rho_2) = 0$ iff ρ_1 and ρ_2 have support on orthogonal subspaces, and $F(\rho_1, \rho_2) = 1$ iff $\rho_1 = \rho_2$.

Thoerem: Monotonicity of fidelity

Suppose $\mathcal{E}$ is a trace preserving operation. Let ρ and σ be density operators. then

$$F\left(\mathcal{E}(\rho), \mathcal{E}(\sigma)\right) \geq F(\rho, \sigma)$$

Remark: Unitary invariance of the fidelity

$$F(U\rho U^{\dagger}, U\sigma U^{\dagger}) = F(\rho, \sigma)$$

Theorem: Strong concativity of the fidelity:

Let $\{p_i\}$ and $\{q_i\}$ be probability distributions over the same index set, and ρ_i and σ_i be density operators, with indices from the same set. Then

$$F\left(\sum_i p_i\rho_i, \sum_i q_i\sigma_i\right) \geq \sum_i \sqrt{p_i\, q_i}\, F(\rho_i, \sigma_i)$$

This property is similar, though not strictly analogous, to the strong convexity of the trace distance.

Remark: Relation between the trace distance and the fidelity:

$$1 - F(\rho, \sigma) \leq D(\rho, \sigma) \leq \sqrt{1 - F(\rho, \sigma)^2}$$

Qualitatively, the trace distance and the fidelity are equivalent measures of distance between quantum states.

Dynamic measures

How well does a quantum channel preserves information?

Example: How well the state $|\psi\rangle$ is preserved by the dephasing channel?

$$F[|\psi\rangle\langle\psi|, \mathcal{E}(|\psi\rangle\langle\psi|)] = \sqrt{\langle\psi|[(1-p)|\psi\rangle\langle\psi| + p\, Z|\psi\rangle\langle\psi|Z]|\psi\rangle}$$
$$= \sqrt{(1-p) + p\, \langle\psi|Z|\psi\rangle^2}$$

The higher the probability of dephasing, the lower the fidelity.

In reality, we do not know the initial state of the system in advance, so we have to quantify the worst case scenario

$$F_{\mathrm{min}}(\mathcal{E}) = \min_{|\psi\rangle} F[|\psi\rangle\langle\psi|, \mathcal{E}(|\psi\rangle\langle\psi|)]$$

For the dephasing channel,

$$F[|\psi\rangle\langle\psi|, \mathcal{E}(|\psi\rangle\langle\psi|)] = \sqrt{(1-p) + p\ \langle\psi|Z|\psi\rangle^2}$$

the second term is non-negative and equals to zero when $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. So the minimal fidelity for this channel is

$$F_{\min}(\mathcal{E}) = \sqrt{1-p}$$

Remark: Allowing mixed states as initial states does not change $F_{\min}$. This is a consequence of the strong concativity

$$F(\rho, \mathcal{E}(\rho)) = F\left[\sum_i \lambda_i |i\rangle\langle i|, \sum_i \lambda_i\ \mathcal{E}(|i\rangle\langle i|)\right] \geq \sum_i \lambda_i\ F[|i\rangle\langle i|, \mathcal{E}(|i\rangle\langle i|)]$$