OPEN QUANTUM SYSTEMS
QUANTUM OPERATIONS

Quantum operations

Quantum operations constitute a theoretical framework for description of the evolution of quantum mechanical systems in the most general circumstances:

$$\rho \quad \rightarrow \quad \rho' = \mathcal{E}(\rho)$$

Examples:

A) Unitary evolution

$$\mathcal{E}(\rho) = U\rho U^\dagger$$

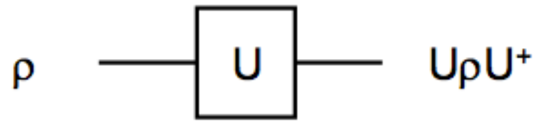
B) Measurement

$$\mathcal{E}_m(\rho) = M_m \rho M_m^\dagger$$

Environment and quantum operations

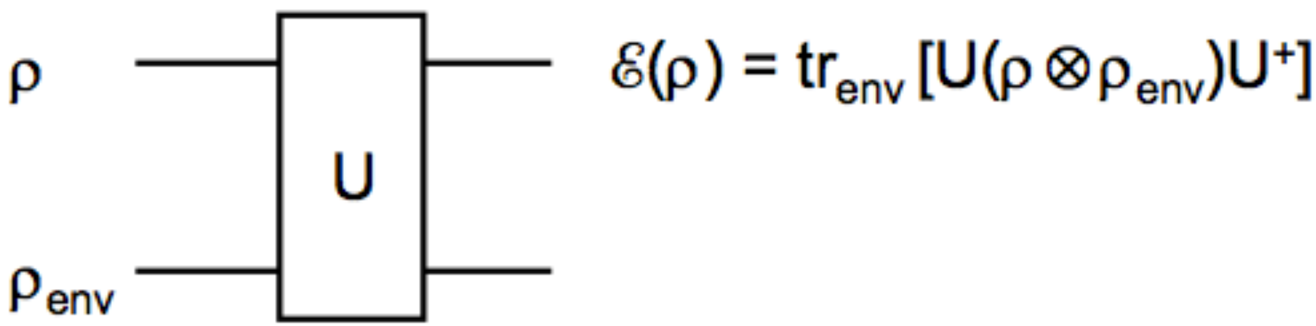
The dynamics of closed quantum systems is unitary:

$$\mathcal{E}(\rho) = U \rho U^\dagger$$



The dynamics of open quantum systems is not unitary in general:
Assume that the initial state of the composite consisting of the system and its environment is a product state $\rho \otimes \rho_{env}$. This composite evolves under the unitary evolution U for a certain duration of time. After then the system decouples from the environment, so we perform the partial trace over the environment to obtain the final state of the system, that is, its reduced density operator:

$$\mathcal{E}(\rho) = \mathrm{tr}_{env} \left[U \left(\rho \otimes \rho_{env} \right) U^\dagger \right]$$



Example:

Assume the system is one qubit in the state ρ, and the environment is one qubit in the initial state $|0\rangle$, and the unitary operation is $CNOT$ with the system as the control:

$$
\begin{aligned}
\mathcal{E}(\rho) &= \mathrm{tr}_{env}\left[U_{CNOT}\,(\rho \otimes |0\rangle\langle 0|)\,U^{\dagger}_{CNOT}\right] \\
&= \mathrm{tr}_{env}\left[(P_0 \otimes I + P_1 \otimes X)\,(\rho \otimes |0\rangle\langle 0|)\,(P_0 \otimes I + P_1 \otimes X)\right] \\
&= \mathrm{tr}_{env}\left[(P_0 \otimes I)\,(\rho \otimes |0\rangle\langle 0|)\,(P_0 \otimes I) + (P_0 \otimes I)\,(\rho \otimes |0\rangle\langle 0|)\,(P_1 \otimes X)\right. \\
&\qquad \left. + \,(P_1 \otimes X)\,(\rho \otimes |0\rangle\langle 0|)\,(P_0 \otimes I) + (P_1 \otimes X)\,(\rho \otimes |0\rangle\langle 0|)\,(P_1 \otimes X)\right] \\
&= \mathrm{tr}_{env}\left[P_0\rho P_0 \otimes |0\rangle\langle 0| + P_0\rho P_1 \otimes |0\rangle\langle 0|X + P_1\rho P_0 \otimes X|0\rangle\langle 0| + P_1\rho P_1 \otimes X|0\rangle\langle 0|X\right] \\
&= \mathrm{tr}_{env}\left[P_0\rho P_0 \otimes |0\rangle\langle 0| + P_0\rho P_1 \otimes |0\rangle\langle 1| + P_1\rho P_0 \otimes |1\rangle\langle 0| + P_1\rho P_1 \otimes |1\rangle\langle 1|\right] \\
&= P_0\rho P_0\,\langle 0|0\rangle + P_0\rho P_1\,\langle 1|0\rangle + P_1\rho P_0\,\langle 0|1\rangle + P_1\rho P_1\,\langle 1|1\rangle \\
&= P_0\,\rho\,P_0 + P_1\,\rho\,P_1
\end{aligned}
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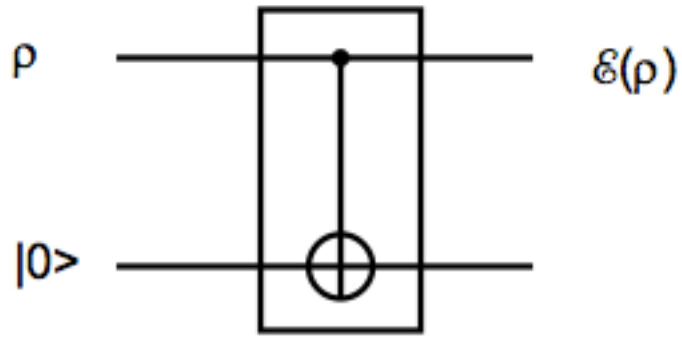


Operator sum representation

Operator sum representation is a representation of quantum operations in terms of the operators on the principal system only:

Let $|e_k\rangle$ be the orthonormal basis for the finite dimensional Hilbert space of the environment, and let $\rho = |e_0\rangle\langle e_0|$ be the initial (pure) state of the environment. Then we can express a quantum operation as

$$\mathcal{E}(\rho) = \mathrm{tr}_{env}\left[U\,(\rho \otimes \rho_{env})\,U^\dagger\right] = \sum_k \langle e_k|U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger|e_k\rangle = \sum_k E_k\,\rho\,E_k^\dagger$$

where $E_k = \langle e_k|U|e_o\rangle$ is an operator on the Hilbert space of the principal system called an *operation element* of the quantum operation.

The operation elements above, $E_k = \langle e_k|U|e_o\rangle$, satisfy the constraint

$$\sum_k E_k^\dagger E_k = I$$

We say that the quantum operation $\mathcal{E}(\rho)$, characterized by these operation elements, is *trace-preserving*.

In general, there exist non-trace-preserving operations for which

$$\sum_k E_k^\dagger E_k \leq I$$

but they describe processes in which an extra information about what occurred in the process is obtained by measurement.

Physical interpretation of operator sum representation

Imagine that a measurement of the environment is performed in the basis $|e_k\rangle$ after the unitary operation U has taken place. By the principle of implicit measurement, such a measurement affects only the state of the environment. Let ρ_k be the state of the principal system given that outcome k occurs, so

$$\begin{aligned}\rho_k &\propto \operatorname{tr}_{env}\Big(|e_k\rangle\langle e_k|U(\rho\otimes|e_0\rangle\langle e_0|)U^\dagger|e_k\rangle\langle e_k|\Big)\\ &= \langle e_k|U(\rho\otimes|e_0\rangle\langle e_0|)U^\dagger|e_k\rangle\\ &= E_k\,\rho\,E_k^\dagger\end{aligned}$$

Normalizing ρ_k gives

$$\rho_k = \frac{E_k\,\rho\,E_k^\dagger}{\mathrm{tr}\left(E_k\,\rho\,E_k^\dagger\right)}$$

where the denominator is the probability that the measurement gives the outcome k

$$p_k = \mathrm{tr}\left(|e_k\rangle\langle e_k|U(\rho\otimes|e_0\rangle\langle e_0|)U^\dagger|e_k\rangle\langle e_k|\right) = \mathrm{tr}\left(E_k\,\rho\,E_k^\dagger\right)$$

Thus

$$\mathcal{E}(\rho) \;=\; \sum_k p_k\,\rho_k = \sum_k E_k\,\rho\,E_k^\dagger$$

$$\mathcal{E}(\rho) \;=\; \sum_k p_k \, \rho_k = \sum_k E_k \, \rho \, E_k^\dagger$$

The action of the quantum operation is equivalent to taking the state ρ and randomly replacing it by

$$\rho_k = \frac{E_k \, \rho \, E_k^\dagger}{\mathrm{tr}\left(E_k \, \rho \, E_k^\dagger\right)}$$

with the probability

$$p_k = \mathrm{tr}\left(E_k \, \rho \, E_k^\dagger\right)$$

Example:

Suppose the states $|e_k\rangle$ are chosen as $|0_E\rangle$ and $|1_E\rangle$. Measurement in the computational basis of the environment qubit does not change the state of the principal system:

$$U_{CNOT} = |0_S 0_E\rangle\langle 0_S 0_E| + |0_S 1_E\rangle\langle 0_S 1_E| + |1_S 1_E\rangle\langle 1_S 0_E| + |1_S 0_E\rangle\langle 1_S 1_E|$$

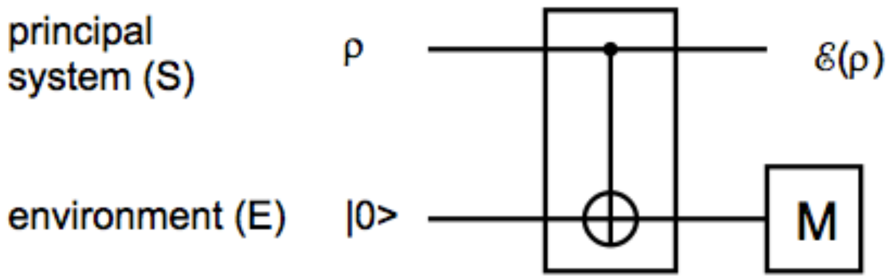
Thus

$$\begin{aligned} E_0 &= \langle 0_E|U_{CNOT}|0_E\rangle = |0_S\rangle\langle 0_S| \\ E_1 &= \langle 1_E|U_{CNOT}|0_E\rangle = |1_S\rangle\langle 1_S| \end{aligned}$$

and therefore

$$\mathcal{E}(\rho) = P_0\, \rho\, P_0 + P_1\, \rho\, P_1$$

This result is in agreement with the result of our previous example.



Measurements and operator sum representation

How do we determine operator sum representation for a given open quantum system?

A) Unitary dynamics

$$E_k = \langle e_k | U | e_0 \rangle$$

B) Measurement on the combined system-environment

Let an initial (product) state $\rho_s \otimes \rho_{env}$ to evolve under the unitary dynamics U and then allow a projective measurement on the combined system-environment.

The final quantum state of the combined system-environment after obtaining the result m of the measurement is

$$\frac{P_m U \left(\rho_s \otimes \rho_{env}\right) U^\dagger P_m}{\mathrm{tr}\left[P_m U \left(\rho_s \otimes \rho_{env}\right) U^\dagger P_m\right]}$$

The final state of the system only is obtained by tracing out the environment

$$\frac{\mathrm{tr}_{env}\left[P_m U \left(\rho_s \otimes \rho_{env}\right) U^\dagger P_m\right]}{\mathrm{tr}\left[P_m U \left(\rho_s \otimes \rho_{env}\right) U^\dagger P_m\right]}$$

Define a map

$$\mathcal{E}_m(\rho) = \mathrm{tr}_{env}\left[P_m U\left(\rho_s \otimes \rho_{env}\right) U^\dagger P_m\right]$$

where $\rho_{env} = \sum_j q_j |j\rangle\langle j|$.

Using an orthonormal basis $|e_k\rangle$ for the environment we get

$$\begin{aligned}\mathcal{E}_m(\rho) &= \sum_{jk} q_j \langle e_k|P_m U\left(\rho_s \otimes |j\rangle\langle j|\right) U^\dagger P_m|e_k\rangle \\ &= \sum_{jk} E_{jk}\, \rho_s\, E_{jk}^\dagger\end{aligned}$$

where

$$E_{jk} = \sqrt{q_j}\, \langle e_k|P_m U|j\rangle$$

System-environment models of any operator sum representation

Given $\{E_k\}$, is there a reasonable model environment and dynamics to produce a quantum operation with the operation elements $\{E_k\}$?

For any quantum operation, trace-preserving or non-trace-preserving, $\mathcal{E}$, with the operation elements $\{E_k\}$, there exists a model environment E, starting in a pure state $|e_0\rangle$, and a model dynamics specified by a unitary operator U and projector P_{env} onto the environment such that

$$\mathcal{E}_m(\rho) = \operatorname{tr}_{env}\left[P_{env} U \left(\rho_s \otimes |e_0\rangle\langle e_0|\right) U^\dagger P_{env}\right]$$

To show this, let us assume that $\mathcal{E}$ is a trace-preserving quantum operation, with operator sum representation generated by operation elements $\{E_k\}$, satisfying the relation $\sum_k E_k^\dagger E_k = I$. In this case, we thus need to find only an appropriate unitary operator U to model the dynamics. Let $|e_k\rangle$ be an orthonormal basis set for environment, with one-to-one correspondence with the index k for E_k. Define an operator U such that

$$U|\psi\rangle|e_0\rangle = \sum_k E_k|\psi\rangle|e_k\rangle$$

where $|e_0\rangle$ is some standard state of the environment. For arbitrary states of the principal system $|\psi\rangle$ and $|\phi\rangle$, $\langle\psi|\langle e_0|U^\dagger U|\phi\rangle|e_0\rangle = \langle\psi|\phi\rangle$, so the operator U acts unitarily on the system-environment state space, and tracing the state over the environment

$$\mathrm{tr}_{env}\left[U\left(\rho_s \otimes |e_0\rangle\langle e_0|\right)U^\dagger\right] = \sum_k E_k\,\rho_s\,E_k^\dagger$$

shows that this model provides a realization of the quantum operation $\mathcal{E}$ with $\{E_k\}$.

Axiomatic approach to quantum operations

Quantum operation $\mathcal{E}$ is defined as a map from the set of density operators of the input space S_1 to the output space S_2, with the following axioms:

(A1) $\mathrm{tr}\,[\mathcal{E}(\rho)]$ is the probability that the process represented by $\mathcal{E}$ occurs, when ρ is the initial state. Thus $0 \leq tr[\mathcal{E}(\rho)] \leq 1$.

(A2) $\mathcal{E}$ is a convex-linear map on the set of density matrices, that is, for probabilities $\{p_i\}$

$$\mathcal{E}\left(\sum_i p_i\,\rho_i\right) = \sum_i p_i\,\mathcal{E}\,(\rho_i)$$

(A3) $\mathcal{E}$ is a *completely positive map*.

That is if $\mathcal{E}$ maps density operators of the system S_1 to the system S_2, then $\mathcal{E}(A)$ must be positive for any positive operator A. Furthermore, if we introduce an extra system R of arbitrary dimensionality, it must be true that $(\mathcal{I} \otimes \mathcal{E})(A)$ is positive for any positive operator A on the combined system RS_1, where $\mathcal{I}$ denotes the identity map on system R.

Theorem: The map $\mathcal{E}$ satisfies axioms A1, A2 and A3 iff

$$\mathcal{E}(\rho) = \sum_k E_k \, \rho \, E_k^\dagger$$

for some set of operators $\{E_k\}$ which map the input Hilbert space to the output Hilbert space, and $\sum_k E_k^\dagger E_k = 1$.

Unitary freedom in operator sum representation

the operation elements is an operator sum representation for a quantum operation are not unique.

Theorem: Suppose $\{E_1, ..., E_m\}$ and $\{F_1, ..., F_n\}$ are operation elements giving rise to quantum operations $\mathcal{E}$ and $\mathcal{F}$, respectively. By appending zero operators to the shorter list of operational elements, we may ensure that $m = n$. Then $\mathcal{E} = \mathcal{F}$ iff there exist complex numbers u_{ij} such that

$$E_i = \sum_j u_{ji} F_j$$

and u is an m-by-m unitary matrix.