MP472 QUANTUM INFORMATION PROCESSING

Introduction to classical and quantum information

Bell inequalities and entanglement

Quantum communication and cryptography, quantum teleportation

Physical and conceptual models of computation and computational complexity classes

Quantum algorithms

Theory of open quantum systems

Quantum error correction

Fault-tolerant quantum computation

Topological quantum computation

Physical implementations

LECTURE NOTES AND REFERENCES

Lecture notes - online access:
http://www.thphys.nuim.ie/Notes/MP472/

Michael Nielsen, Isaac Chuang
Quantum Computation and Quantum Information
Cambridge University Press, 2000

John Preskill
Lecture Notes - Physics 219, Caltech
http://www.theory.caltech.edu/ preskill/ph219/index.html

REQUIREMENTS

The total mark consists of:

Examination: constitutes 80% of the total mark
duration: 120 minutes,
maximum mark: 100 points.

Continuous Assessment: 20% of the total mark
homework assignments, quizzes.

MP472 QUANTUM INFORMATION PROCESSING

Classical information
- classical bit
- Boolean function
- Boolean circuit

Quantum information
- quantum bit(s)
- quantum operations
- quantum state measurement
- quantum circuit
- example: quantum entangler

Example of quantum information processing:
Teleportation

Classical information and its processing

An elementary unit of classical information is **bit**

$$\mathbb{B} = \{0, 1\}$$

Information is physical (Rolf Landauer, IBM):

The values 0 and 1 of the bit correspond to two distinct values (states) of some physical quantity, for example electric voltage.

A **Boolean function** on n variables

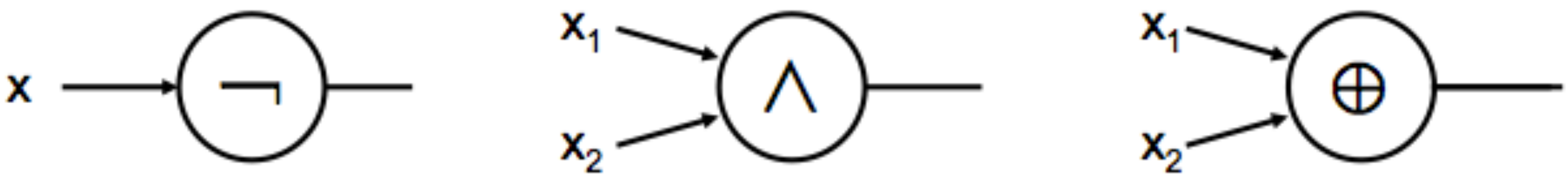
$$F(x_1, x_2, ..., x_n) : \mathbb{B}^n \to \mathbb{B}^k$$

Examples: simple Boolean functions

negation $\neg x$ (bit flip)

conjunction $x_1 \wedge x_2$

addition mod(2) $x_1 \oplus x_2$



x		$\neg x$
0	→	1
1	→	0

(bit flip)

x_1	x_2		$x_1 \wedge x_2$
0	0	→	0
0	1	→	0
1	0	→	0
1	1	→	1

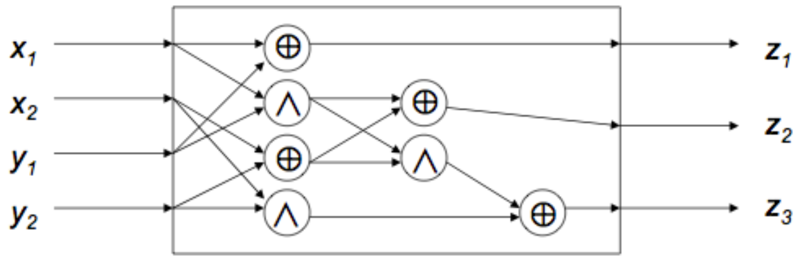
x_1	x_2		$x_1 \oplus x_2$
0	0	→	0
0	1	→	1
1	0	→	1
1	1	→	0

A **Boolean circuit** is a representation of a Boolean function as a composition of other Boolean functions from a set $\mathcal{B}$, for example:

$$\mathcal{B}\{\wedge, \oplus\}$$

A circuit over $\mathcal{B}$ is a sequence of assignments involving n input variables $\{x_1, x_2, ..., x_n\}$ and several auxiliary variables $\{y_1, y_2, ..., y_k\}$ where $y_k = f_k(u_1, ..., u_r)$ and each of the variables $u_1, ..., u_r$ are either input variables or auxiliary variables preceeding y_k.

Example: Addition of two 2-digit numbers (Kitaev et al.)



A basis $\mathcal{B}$ is called complete , if for any Boolean function f, there is a circuit over $\mathcal{B}$ that computes f. For example $\mathcal{B}\{\wedge, \oplus\}$.

Quantum information

Quantum bit or **qubit** is a two dimensional Hilbert space $\mathcal{H}^2 \simeq \mathbb{C}^2$.

Qubit values are vectors, states, from this Hilbert space:

$$|\phi\rangle = c_0|0\rangle + c_1|1\rangle$$

where $|0\rangle$ and $|1\rangle$ are an orthonormal set called the **standard computational basis** and $c_0, c_1 \in \mathbb{C}$ and $|c_0|^2 + |c_1|^2 = 1$.

Physical realization of a qubit can for example be a spin-1/2 particle:

$$|0\rangle = |\uparrow\rangle \quad |1\rangle = |\downarrow\rangle \qquad |\phi\rangle = c_{\uparrow}|\uparrow\rangle + c_{\downarrow}|\downarrow\rangle$$

or two energy levels of an atom or ion,
or opposite superconducting fluxes in a superconducting flux qubit,
or ...

Quantum logic operations are rotations of a quantum state vector in a Hilbert space:

they are **unitary**, and thus **reversible**, operations.

(Classical computation can be made reversible.)

Qubits

A quantum state of n qubits is a vector in 2^n-dimensional Hilbert space:

$$\bigotimes_{k=1}^{n} \mathcal{H}^2 = \mathcal{H}^2 \otimes \mathcal{H}^2 \otimes ...\mathcal{H}^2 \quad \text{(n-times)} = \mathcal{H}^{2^n}$$

Examples: Composite product states

$$|\phi\rangle \;=\; |0\rangle \otimes |0\rangle = |0\rangle|0\rangle = |00\rangle$$

or

$$|\psi\rangle \;=\; (c_0|0\rangle + c_1|1\rangle) \otimes |0\rangle = c_{00}|00\rangle + c_{10}|10\rangle$$

where in the latter we identified $c_{00} = c_0$ and $c_{10} = c_1$.

Examples: Entangled states: the Bell states

$$
\begin{aligned}
|\beta_{00}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
|\beta_{01}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
|\beta_{10}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
|\beta_{11}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)
\end{aligned}
$$

No-cloning theorem: Quantum information can not be cloned (copied):

Assume there is a cloning operator $\hat{C}$ such that

$$\hat{C}|0\rangle = |0\rangle \otimes |0\rangle = |00\rangle \quad \text{and} \quad \hat{C}|1\rangle = |1\rangle \otimes |1\rangle = |11\rangle$$

then applying it onto a superposition $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ proofs the theorem

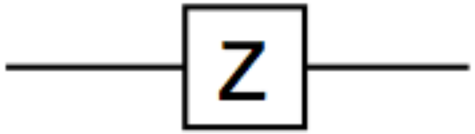
$$\hat{C}|\psi\rangle \neq |\psi\rangle \otimes |\psi\rangle$$

$$\begin{aligned}
\hat{C}\left(c_0|0\rangle + c_1|1\rangle\right) &= c_0\hat{C}|0\rangle + c_1\hat{C}|1\rangle \\
&= c_0|00\rangle + c_1|11\rangle \\
&\neq \left(c_0|0\rangle + c_1|1\rangle\right) \otimes \left(c_0|0\rangle + c_1|1\rangle\right) \\
&= c_0^2|00\rangle + c_0c_1|01\rangle + c_0c_1|10\rangle + c_1^2|11\rangle
\end{aligned}$$

Quantum computing operations

Single qubit gates

Phase flip $\hat{Z}$



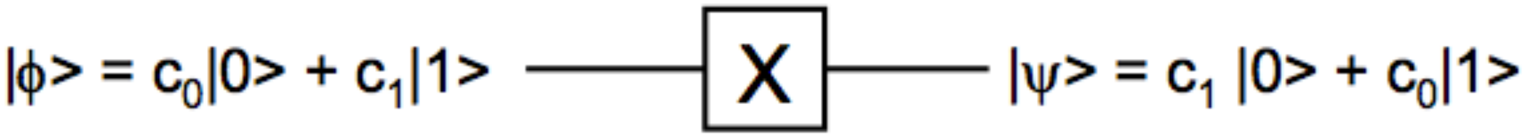
$$\begin{aligned} \hat{Z}|0\rangle &= |0\rangle \\ \hat{Z}|1\rangle &= -|1\rangle \\ \hat{Z}\,(c_0|0\rangle + c_1|1\rangle) &= c_0\,\hat{Z}\,|0\rangle + c_1\,\hat{Z}\,|1\rangle = c_0|0\rangle - c_1|1\rangle \end{aligned}$$

This operation or gate has no analog in classical world.

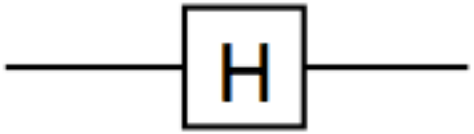
Homework:
Show that the states $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|\hat{Z}\psi\rangle = \hat{Z}|\psi\rangle$ are orthogonal.

Bit flip $\hat{X}$

$$
\begin{aligned}
\hat{X}|0\rangle &= |1\rangle \\
\hat{X}|1\rangle &= |0\rangle \\
\hat{X}\,(c_0|0\rangle + c_1|1\rangle) &= c_0\,\hat{X}\,|0\rangle + c_1\,\hat{X}\,|1\rangle = c_1|0\rangle + c_0|1\rangle
\end{aligned}
$$



Hadamard gate $\hat{H}$



$$
\begin{aligned}
\hat{H}|0\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \\
\hat{H}|1\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \\
\hat{H}\left(c_0|0\rangle + c_1|1\rangle\right) &= c_0\,\hat{H}\,|0\rangle + c_1\,\hat{H}\,|1\rangle \\
&= \frac{c_0}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) + \frac{c_1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \\
&= \frac{c_0 + c_1}{\sqrt{2}}|0\rangle + \frac{c_0 - c_1}{\sqrt{2}}|1\rangle
\end{aligned}
$$

<u>Homework:</u>

Show what operations correspond to the following products $\hat{H}\hat{H}$, $\hat{H}\hat{Z}\hat{H}$, $\hat{H}\hat{X}\hat{H}$.

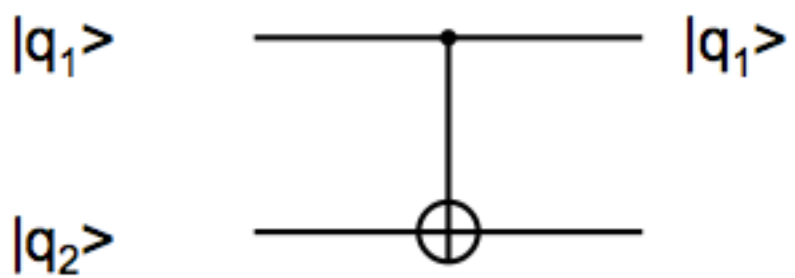
Two-qubit gates

Two-qubit states have the standard computational basis $\mathcal{B} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

Controlled-NOT

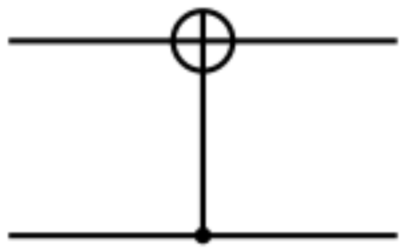
$CNOT_{12}$ (the first qubit is the control qubit, the second is the target qubit):

$$\begin{aligned} CNOT_{12}|00\rangle &= |00\rangle \\ CNOT_{12}|01\rangle &= |01\rangle \\ CNOT_{12}|10\rangle &= |11\rangle \\ CNOT_{12}|11\rangle &= |10\rangle \\ CNOT_{12}\left(c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle\right) &= c_{00}|00\rangle + c_{01}|01\rangle + c_{11}|10\rangle + c_{10}|11\rangle \end{aligned}$$



$CNOT_{21}$:

$$\begin{aligned}
CNOT_{21}|00\rangle &= |00\rangle \\
CNOT_{21}|01\rangle &= |11\rangle \\
CNOT_{21}|10\rangle &= |10\rangle \\
CNOT_{21}|11\rangle &= |01\rangle \\
CNOT_{21}\left(c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle\right) &= c_{00}|00\rangle + c_{11}|01\rangle + c_{10}|10\rangle + c_{01}|11\rangle
\end{aligned}$$



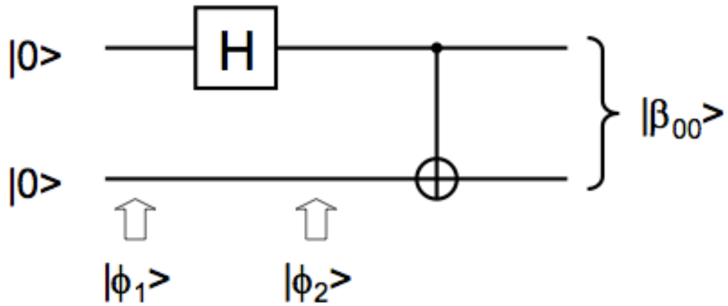
Application: **the Bell state generator**

$$|\phi_1\rangle = |0\rangle \otimes |0\rangle = |00\rangle$$

$$|\phi_2\rangle = \left(\hat{H} \otimes \hat{1}\right)|\phi_1\rangle = \hat{H}|0\rangle \otimes \hat{1}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

$$|\beta_{00}\rangle = CNOT|\phi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

(Here $CNOT$ stays for $CNOT_{12}$.)



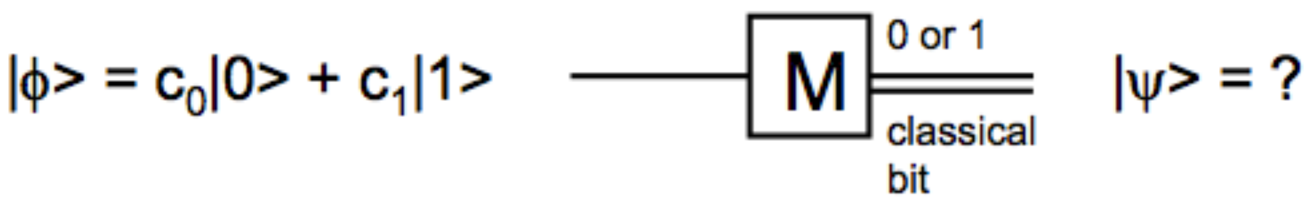
Homework: Design circuits to generate the other Bell states

$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \qquad |\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \qquad |\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Single qubit measurement

Measurement of one qubit $|\phi\rangle = c_0|0\rangle + c_1|1\rangle$ in the standard computational basis gives classical bit of information:

- with the probability $|c_0|^2$ the measurement gives the result $M = 0$ and the quantum state immediately after the measurement has collapsed to $|\psi\rangle = |0\rangle$;
- with the probability $|c_1|^2$ the measurement gives the result $M = 1$ and the quantum state immediately after the measurement has collapsed to $|\psi\rangle = |1\rangle$.



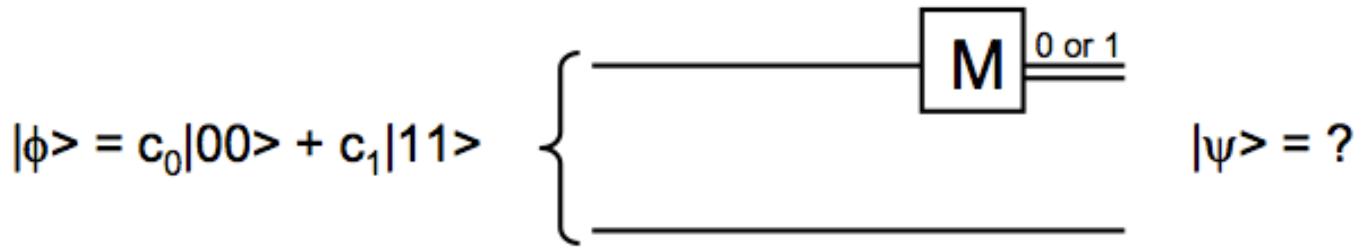
Measurement of an entangled state - Einstein-Podolsky-Rosen paradox

Measurement of the first qubit of a two-qubit entangled state $|\phi\rangle = c_{00}|00\rangle + c_{11}|11\rangle$ yields the following outcome:

- with the probability $\left|c_{00}\right|^2$ the measurement gives the result $M_1 = 0$ and the quantum state immediately after the measurement has collapsed to $|\psi\rangle = |00\rangle$;

- with the probability $|c_{11}|^2$ the measurement gives the result $M_1 = 1$ and the quantum state immediately after the measurement has collapsed to $|\psi\rangle = |11\rangle$.

The measurement of one qubit of an entangled two-qubit state completely determines the state of the other qubit after the measurement even if both qubits are spatially separated and can not communicate or interact.

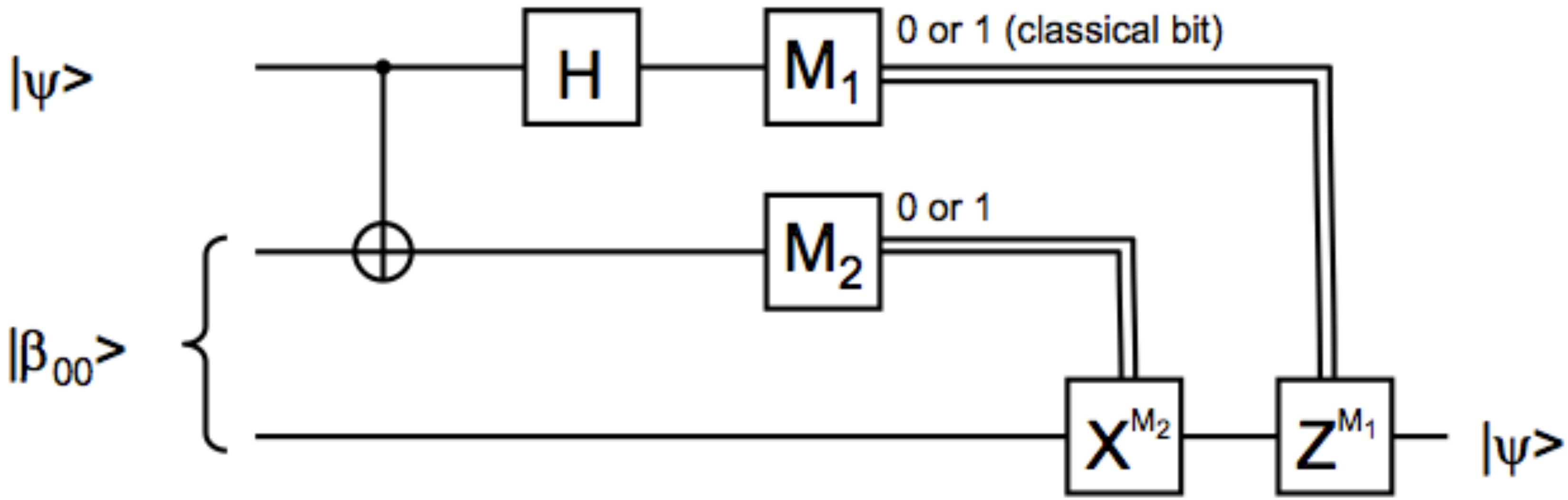


No classical analog !!!

Teleportation

Teleportation

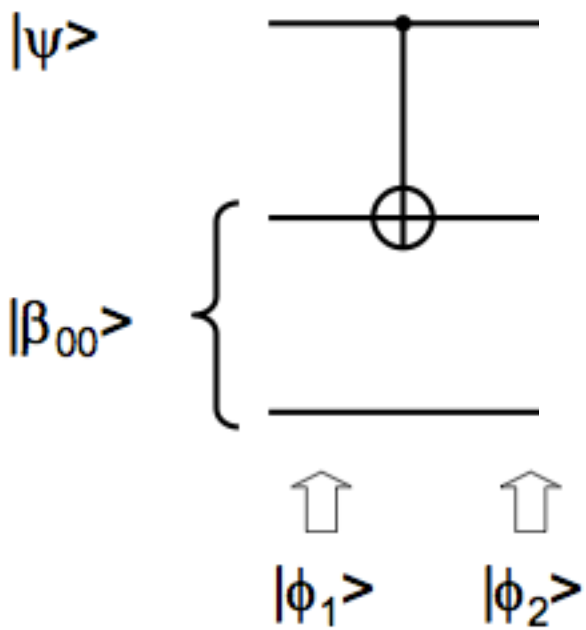
Teleport an unknown qubit state $|\psi\rangle$ using the Bell state $|\beta_{00}\rangle$ and single-qubit and two-qubit operations, two measurements and communication of two classical bits.



$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle \qquad |\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

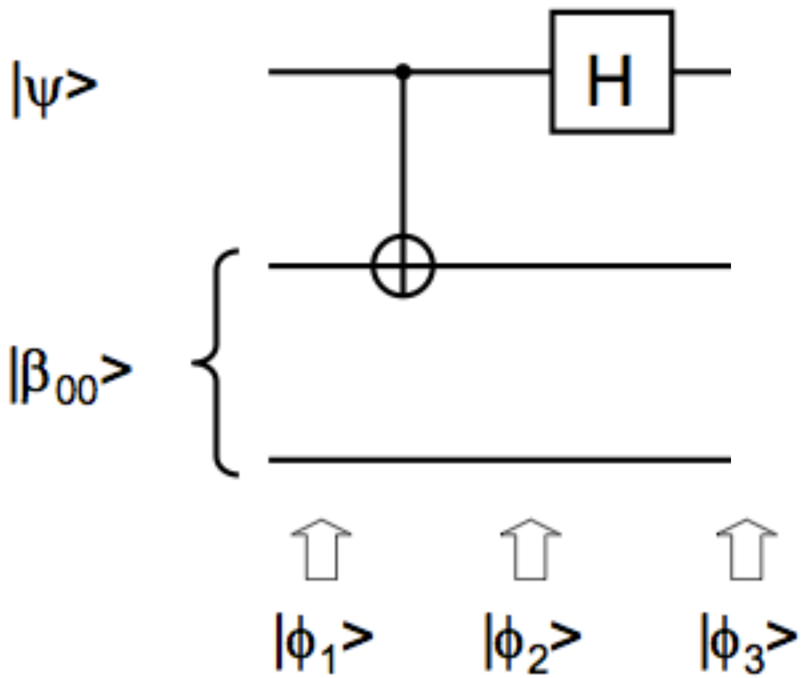
$$|\phi_1\rangle = |\psi\rangle \otimes |\beta_{00}\rangle = |\psi\rangle|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(c_0|000\rangle + c_0|011\rangle + c_1|100\rangle + c_1|111\rangle)$$

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}(c_0|000\rangle + c_0|011\rangle + c_1|110\rangle + c_1|101\rangle)$$



$$|\phi_2\rangle = \frac{1}{\sqrt{2}}(c_0|000\rangle + c_0|011\rangle + c_1|110\rangle + c_1|101\rangle)$$

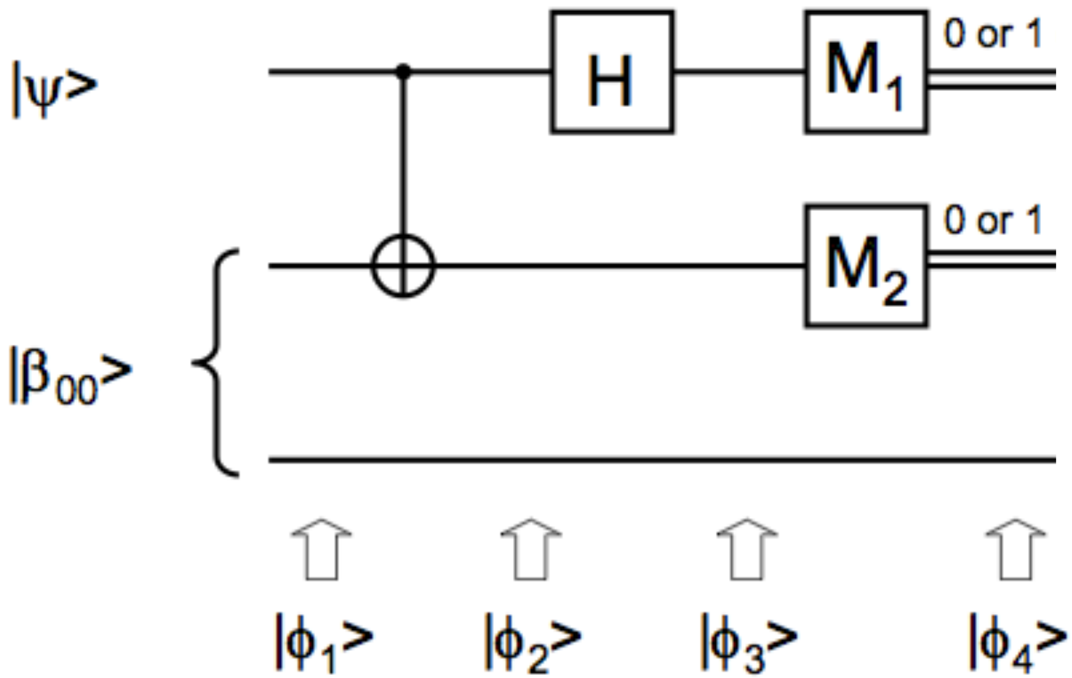
$$|\phi_3\rangle = \frac{1}{2}(c_0|000\rangle + c_0|100\rangle + c_0|011\rangle + c_0|111\rangle + c_1|010\rangle - c_1|110\rangle + c_1|001\rangle - c_1|101\rangle)$$



$$|\phi_3\rangle \;=\; \frac{1}{2}(c_0|000\rangle + c_0|100\rangle + c_0|011\rangle + c_0|111\rangle + c_1|010\rangle - c_1|110\rangle + c_1|001\rangle - c_1|101\rangle)$$

Four possible results of the measurements on the first and second qubit are

$$(i)\quad M_1 = 0,\quad M_2 = 0 \qquad (ii)\quad M_1 = 0,\quad M_2 = 1$$
$$(iii)\quad M_1 = 1,\quad M_2 = 0 \qquad (iv)\quad M_1 = 1,\quad M_2 = 1$$

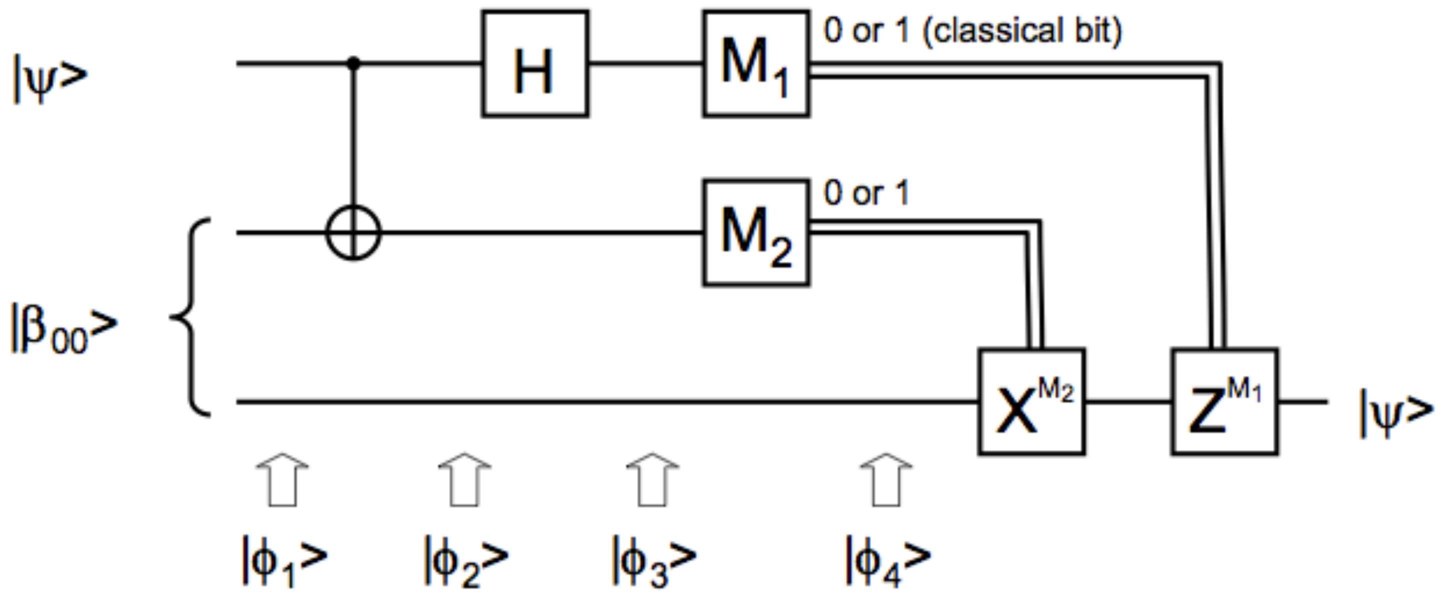


$$|\phi_3\rangle = \frac{1}{2}(c_0|000\rangle + c_0|100\rangle + c_0|011\rangle + c_0|111\rangle + c_1|010\rangle - c_1|110\rangle + c_1|001\rangle - c_1|101\rangle)$$

Measurement results: $M_1 = 0, \quad M_2 = 0$

$$\begin{aligned} |\phi_4\rangle &= c_0|000\rangle + c_1|001\rangle = |00\rangle \otimes (c_0|0\rangle + c_1|1\rangle) = |00\rangle \otimes |\psi'\rangle \\ |\psi\rangle &= \hat{Z}^0 \, \hat{X}^0 \, |\psi'\rangle = |\psi'\rangle = c_0|0\rangle + c_1|1\rangle \end{aligned}$$

The final state of the third qubit is now the same as the initial state of the first qubit.

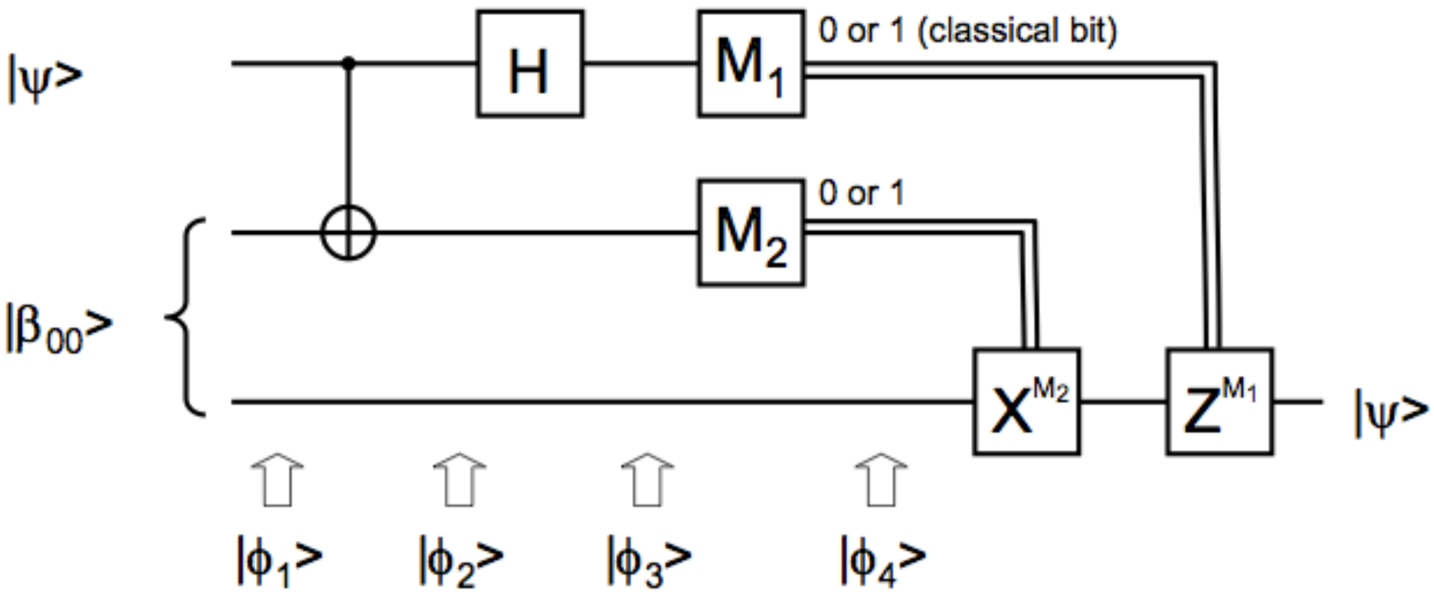


$$|\phi_3\rangle = \frac{1}{2}(c_0|000\rangle + c_0|100\rangle + c_0|011\rangle + c_0|111\rangle + c_1|010\rangle - c_1|110\rangle + c_1|001\rangle - c_1|101\rangle)$$

Measurement results: $M_1 = 1, \quad M_2 = 0$

$$|\phi_4\rangle = c_0|100\rangle - c_1|101\rangle = |10\rangle \otimes (c_0|0\rangle - c_1|1\rangle) = |10\rangle \otimes |\psi'\rangle$$
$$|\psi\rangle = \hat{Z}^1\ \hat{X}^0\ |\psi'\rangle = \hat{Z}\,(c_0|0\rangle - c_1|1\rangle) = c_0|0\rangle + c_1|1\rangle$$

The final state of the third qubit is now the same as the initial state of the first qubit.

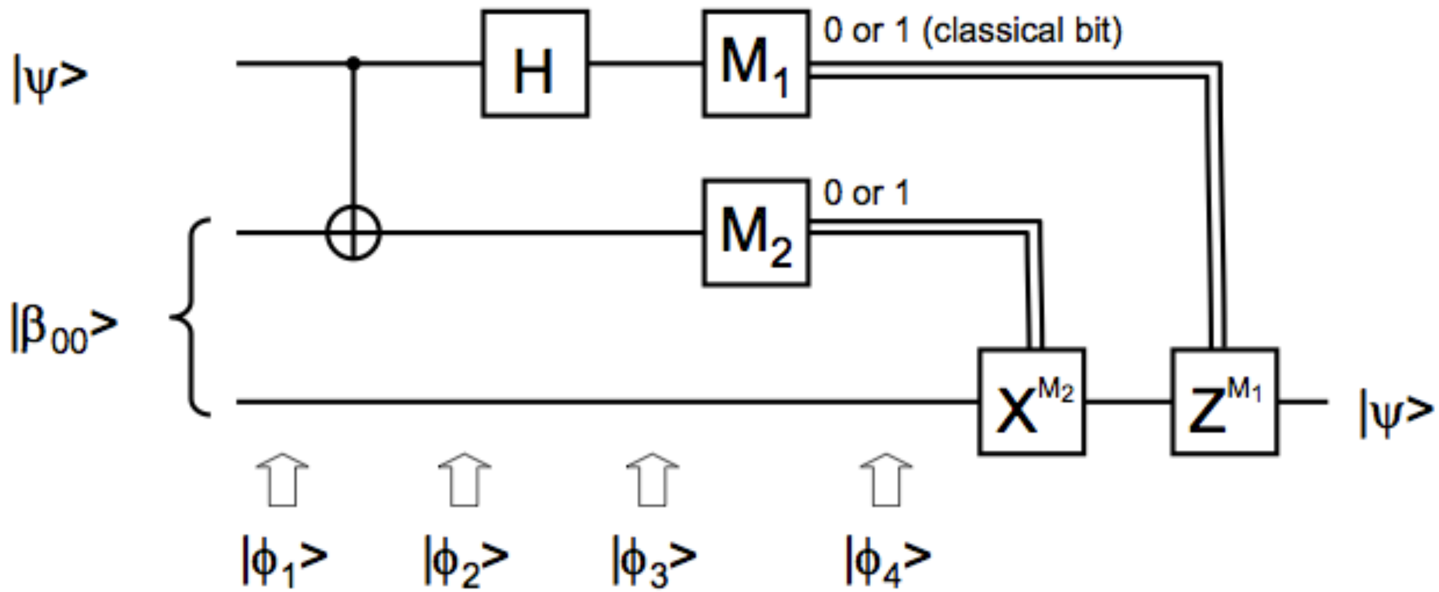


$$|\phi_3\rangle = \frac{1}{2}(c_0|000\rangle + c_0|100\rangle + c_0|011\rangle + c_0|111\rangle + c_1|010\rangle - c_1|110\rangle + c_1|001\rangle - c_1|101\rangle)$$

Measurement results: $M_1 = 0, \quad M_2 = 1$

$$|\phi_4\rangle = c_0|011\rangle + c_1|010\rangle = |01\rangle \otimes (c_1|0\rangle + c_0|1\rangle) = |01\rangle \otimes |\psi'\rangle$$
$$|\psi\rangle = \hat{Z}^0\,\hat{X}^1\,|\psi'\rangle = \hat{X}\,(c_1|0\rangle + c_0|1\rangle) = c_0|0\rangle + c_1|1\rangle$$

The final state of the third qubit is now the same as the initial state of the first qubit.

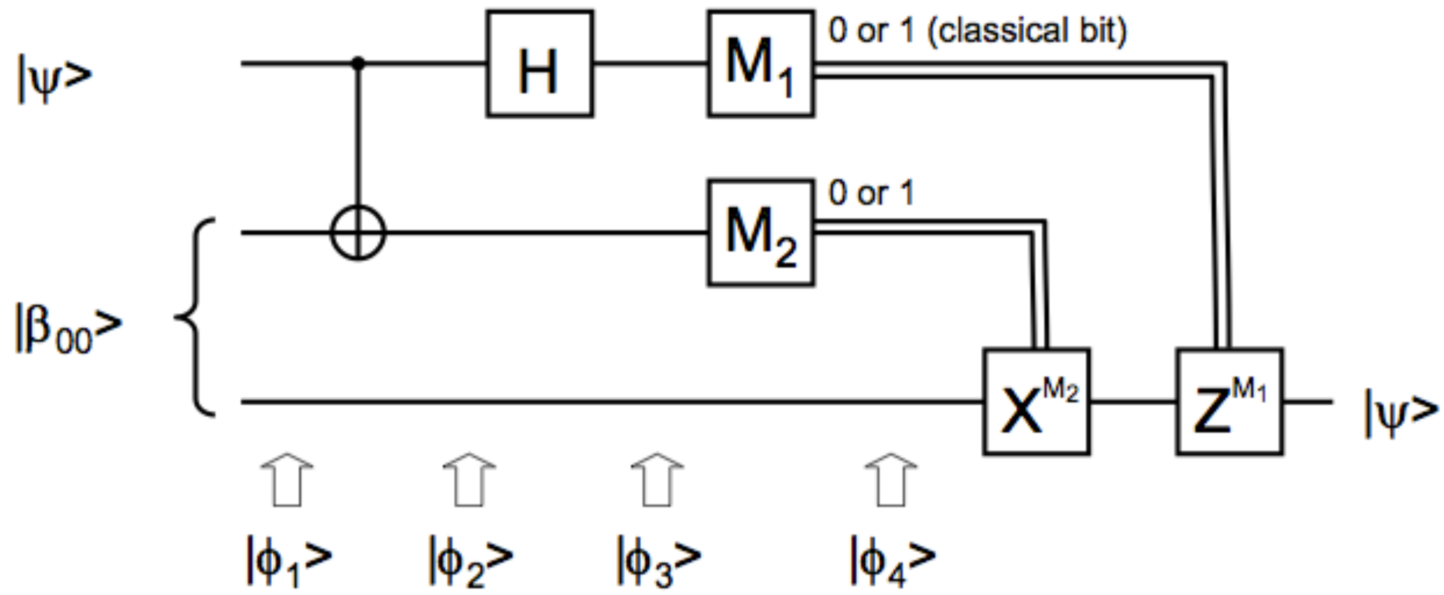


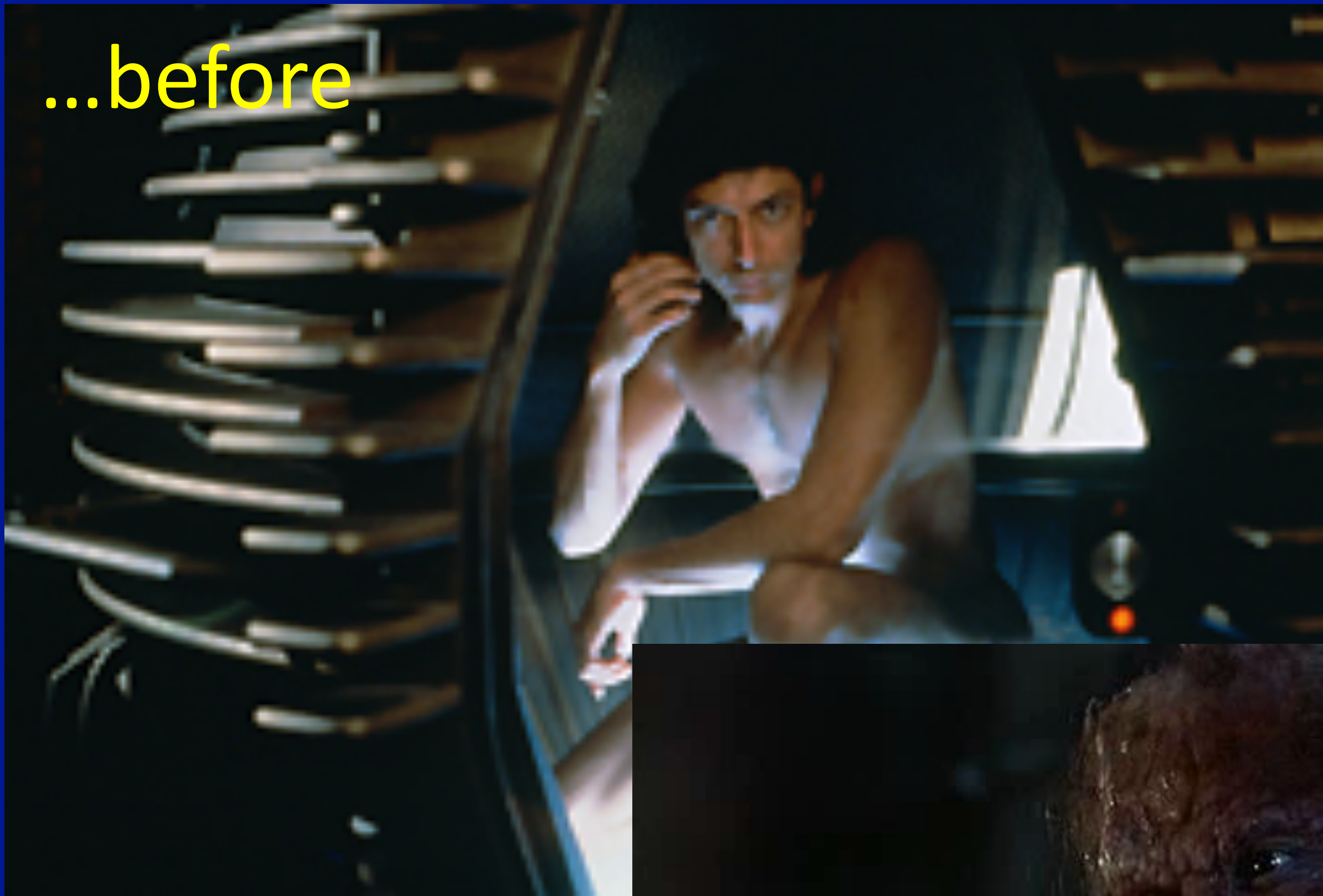
$$|\phi_3\rangle = \frac{1}{2}(c_0|000\rangle + c_0|100\rangle + c_0|011\rangle + c_0|111\rangle + c_1|010\rangle - c_1|110\rangle + c_1|001\rangle - c_1|101\rangle)$$

Measurement results: $M_1 = 1, \quad M_2 = 1$

$$|\phi_4\rangle = c_0|111\rangle - c_1|110\rangle = |11\rangle \otimes (-c_1|0\rangle + c_0|1\rangle) = |11\rangle \otimes |\psi'\rangle$$
$$|\psi\rangle = \hat{Z}^1 \, \hat{X}^1 \, |\psi'\rangle = \hat{Z}\hat{X}\,(-c_1|0\rangle + c_0|1\rangle) = \hat{Z}\,(c_0|0\rangle - c_1|1\rangle) = c_0|0\rangle + c_1|1\rangle$$

The final state of the third qubit is now the same as the initial state of the first qubit.





You better learn
how teleportation works!!!