MP465 – Advanced Electromagnetism

Solutions to Problem Set 5

1. (a) This is straightforward if we use the following: if the current density has the form $\vec{J}(t,\vec{r}) = \mathrm{Re}[\tilde{\vec{J}}(\vec{r})e^{-i\omega t}]$, then the electric dipole moment is $\vec{p}(t) = \mathrm{Re}[\tilde{\vec{p}}_0 e^{-i\omega t}]$ where

$$\tilde{\vec{p}}_0 = \frac{i}{\omega}\int \tilde{\vec{J}}(\vec{r})\,\mathrm{d}^3\vec{r}.$$

To use this, we need $\tilde{\vec{J}}$, but we've seen how to get this: since $\tilde{\vec{J}}$ is a complex vector, we can write it in terms of its real and imaginary parts $\vec{J}_R$ and $\vec{J}_I$ as $\tilde{\vec{J}} = \vec{J}_R + i\vec{J}_I$, and we've shown before that doing this gives $\vec{J} = \vec{J}_R \cos\omega t + \vec{J}_I \sin\omega t$. For the current density given, this means that $\vec{J}_R = \vec{0}$ and $\vec{J}_I = I_0(1-|y|/L)\delta(x)\delta(z)\hat{e}_y$ and thus

$$\tilde{\vec{J}}(\vec{r}) = iI_0\left(1-\frac{|y|}{L}\right)\delta(x)\delta(z)\hat{e}_y$$

and thus

$$\begin{aligned}
\tilde{\vec{p}}_0 &= \frac{i}{\omega}\int iI_0\left(1-\frac{|y|}{L}\right)\delta(x)\delta(z)\hat{e}_y\,\mathrm{d}^3\vec{r} \\
&= -\frac{I_0\hat{e}_y}{\omega}\int\left(1-\frac{|y|}{L}\right)\delta(x)\delta(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= -\frac{I_0\hat{e}_y}{\omega}\int_{-L}^{L}\left(1-\frac{|y|}{L}\right)\mathrm{d}y \\
&= -\frac{2I_0\hat{e}_y}{\omega}\int_{0}^{L}\left(1-\frac{y}{L}\right)\mathrm{d}y
\end{aligned}$$

where in the final step we've used the fact that the integrand is an even function of y and $|y| = y$ if $y > 0$. The integral is easily done, with the result being $\tilde{\vec{p}}_0 = -I_0 L\hat{e}_y/\omega$.

(b) In the far-zone approximation, the magnetic field amplitude is

$$\tilde{\vec{B}}(\vec{r}) \approx \frac{\mu_0\omega^2}{4\pi c}\frac{\hat{e}_r \times \tilde{\vec{p}}_0}{r}e^{ikr}$$

and since $\tilde{\vec{E}} \approx c\tilde{\vec{B}} \times \hat{e}_r$, the time-averaged Poynting vector is

$$\begin{aligned}\langle\vec{S}\rangle &= \frac{1}{2\mu_0}\tilde{\vec{E}} \times \tilde{\vec{B}}^* \\ &\approx \frac{\mu_0\omega^4}{32\pi^2 c}\frac{|\hat{e}_r \times \tilde{\vec{p}}_0|^2}{r^2}\hat{e}_r\end{aligned}$$

and since the surface area element of a sphere of radius r is $\mathrm{d}\vec{\sigma} = r^2\mathrm{d}\Omega\,\hat{e}_r$, the power distribution is

$$\begin{aligned}\frac{\mathrm{d}\bar{P}}{\mathrm{d}\Omega} &= \frac{\langle\vec{S}\rangle\cdot\mathrm{d}\vec{\sigma}}{\mathrm{d}\Omega} \\ &\approx \frac{\mu_0\omega^4|\hat{e}_r \times \tilde{\vec{p}}_0|^2}{32\pi^2 c}.\end{aligned}$$

We have the dipole amplitude, so we can compute the cross-product:

$$\begin{aligned}\hat{e}_r \times \tilde{\vec{p}}_0 &= (\sin\theta\cos\phi\,\hat{e}_x + \sin\theta\sin\phi\,\hat{e}_y + \cos\theta\,\hat{e}_z) \times (-\frac{I_0 L}{\omega}\hat{e}_y) \\ &= \frac{I_0 L}{\omega}(\cos\theta\,\hat{e}_x - \sin\theta\cos\phi\,\hat{e}_z)\end{aligned}$$

and the norm-squared of this is $I_0^2L^2(\cos^2\theta + \sin^2\theta\cos^2\phi)/\omega^2$. We could leave it like this, but I'm going to use $\cos^2\theta = 1 - \sin^2\theta$ and $\cos^2\phi = 1 - \sin^2\phi$ to turn it into the slightly simpler-looking $1 - \sin^2\theta\sin^2\phi$ to get

$$\frac{\mathrm{d}\bar{P}}{\mathrm{d}\Omega} \approx \frac{\mu_0 I_0^2 L^2\omega^2}{32\pi^2 c}\left(1 - \sin^2\theta\sin^2\phi\right).$$

2. (a) We use the continuity equation to prove this. The definition of the electric dipole moment is $\vec{p} = \int \rho\vec{r}\,\mathrm{d}^3\vec{r}$, so the time-derivative

of the i^{th}-component is

$$\begin{aligned}
\dot{p}_i(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int \rho(t,\vec{r})x_i\,\mathrm{d}^3\vec{r} \\
&= \int \frac{\partial\rho}{\partial t}(t,\vec{r})x_i\,\mathrm{d}^3\vec{r} \\
&= \int \left(-\vec{\nabla}\cdot\vec{J}(t,\vec{r})\right)x_i\,\mathrm{d}^3\vec{r} \\
&= -\int \left[\vec{\nabla}\cdot\left(\vec{J}(t,\vec{r})x_i\right) - \vec{J}(t,\vec{r})\cdot\vec{\nabla}x_i\right]\mathrm{d}^3\vec{r} \\
&= -\oint_\Sigma \vec{J}(t,\vec{r})x_i\cdot\mathrm{d}\vec{\sigma} + \int \vec{J}(t,\vec{r})\cdot\hat{e}_i\,\mathrm{d}^3\vec{r}
\end{aligned}$$

where Σ is the "surface at infinity" and we've used $\vec{\nabla}x_i = \hat{e}_i$. Making the usual assumption that all sources are zero at infinity, the surface integral vanishes and the integrand of the second integral is $J_i(t,\vec{r})$, and thus we see

$$\dot{\vec{p}}(t) = \int \vec{J}(t,\vec{r})\,\mathrm{d}^3\vec{r}.$$

(b) Suppose that at time $t=0$ the charge is at position $\vec{r}_0$, has velocity $\vec{u}_0$ and undergoes a constant acceleration $\vec{a}$. We know from elementary mechanics that its position at time t is

$$\vec{r}(t) = \vec{r}_0 + \vec{u}_0 t + \frac{1}{2}\vec{a}t^2.$$

We know that for n point charges $q_1,\ldots,q_n$ located at positions $\vec{r}_1,\ldots,\vec{r}_n$, the electric dipole moment is

$$\vec{p} = \sum_{i=1}^{n} q_i\vec{r}_i$$

and thus our single point charge at $\vec{r}(t)$ has

$$\vec{p}(t) = q\left(\vec{r}_0 + \vec{u}_0 t + \frac{1}{2}\vec{a}t^2\right).$$

Two time derivatives yields $\ddot{\vec{p}}(t) = q\vec{a}$, so the far-zone magnetic field is

$$\begin{aligned}\vec{B}(t,\vec{r}) &\approx \frac{\mu_0}{4\pi c}\frac{\ddot{\vec{p}}(t-r/c)\times\hat{e}_r}{r}\\ &= \frac{\mu_0}{4\pi c}\frac{q\vec{a}\times\hat{e}_r}{r}.\end{aligned}$$

The far-zone electric field is

$$\vec{E}(t,\vec{r}) \approx c\vec{B}(t,\vec{r})\times\hat{e}_r$$

so the far-zone Poynting vector (*not* time-averaged) is

$$\begin{aligned}\vec{S} &= \frac{1}{\mu_0}\vec{E}\times\vec{B}\\ &\approx \frac{c}{\mu_0}\left(\vec{B}\times\hat{e}_r\right)\times\vec{B}\\ &\approx \frac{c}{\mu_0}|\vec{B}|^2\hat{e}_r\end{aligned}$$

since $\vec{B}$ and $\hat{e}_r$ are normal to one another. Thus, we find

$$\begin{aligned}\vec{S} &\approx \frac{\mu_0 q^2}{16\pi^2 c}\frac{|\vec{a}\times\hat{e}_r|^2}{r^2}\,\hat{e}_r\\ &= \frac{q^2}{16\pi^2\epsilon_0 c^3}\frac{|\vec{a}\times\hat{e}_r|^2}{r^2}\,\hat{e}_r\end{aligned}$$

where we've followed the convention of expressing μ_0 as $1/\epsilon_0 c^2$ when our sources are electric in nature. The total power radiated by this charge through a sphere of very large radius r is therefore

$$\begin{aligned}P &= \int \vec{S}\cdot d\vec{\sigma}\\ &\approx \int\left(\frac{q^2}{16\pi^2\epsilon_0 c^3}\frac{|\vec{a}\times\hat{e}_r|^2}{r^2}\,\hat{e}_r\right)\cdot\left(r^2 d\Omega\,\hat{e}_r\right)\\ &= \frac{q^2}{16\pi^2\epsilon_0 c^3}\int|\vec{a}\times\hat{e}_r|^2 d\Omega.\end{aligned}$$

This is as far as we can go without explicitly introducing a coordinate system. We're free to choose the most convenient one, so pick one where the positive z-axis is aligned with the acceleration $\vec{a}$. This means that the angle between $\vec{a}$ and $\hat{e}_r$ is the spherical coordinate angle θ, and so $|\vec{a} \times \hat{e}_r| = a \sin\theta$, where $a = |\vec{a}|$. In spherical coordinates, the solid angle element is $\mathrm{d}\Omega = \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$, and since we're integrating over the entire sphere,

$$\begin{aligned} P &\approx \frac{q^2}{16\pi^2\epsilon_0 c^3} \int (a^2 \sin^2\theta)(\sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi) \\ &= \frac{q^2 a^2}{16\pi^2\epsilon_0 c^3} \left(\int_0^\pi \sin^3\theta \, \mathrm{d}\theta \right) \left(\int_0^{2\pi} \mathrm{d}\phi \right). \end{aligned}$$

The first integral is $4/3$ and the second is 2π, so we end up with a total radiated power of

$$P \approx \frac{q^2 a^2}{6\pi\epsilon_0 c^3}.$$

Note this depends on a^2, so the same power is radiated regardless of if the charge is *ac*celerating or *de*celerating.

A comment: our form for $\vec{r}(t)$ is a nonrelativistic formula, so the above isn't quite correct for an extremely fast-moving charge. Thus, if we wanted to find the power lost as a charged particle falls into a black hole, for example, we'd have to do a more sophisticated calculation beyond the scope of this module. But it can be done.

3. The brute-force way to do this is to use $\vec{v} = v_y \hat{e}_y + v_z \hat{e}_z$, do all the cross- and dot-products and solve the equations, but here's a slightly easier way to do it...

   Since the parallel component points in the same direction as the velocity, this means $\vec{E} \cdot \vec{v} = E_\parallel v$ and $\vec{B} \cdot \vec{v} = B_\parallel v$ where $v = |\vec{v}|$. Thus, the transformation laws give $E'_\parallel v = \vec{E} \cdot \vec{v}$ and $B'_\parallel v = \vec{B} \cdot \vec{v}$. But since $\vec{B}' = -\lambda \vec{E}'/c$, their parallel components have the same relation and thus $B'_\parallel v = -\lambda E'_\parallel v/c$. The upshot is that $\vec{B} \cdot \vec{v} = -\lambda \vec{E} \cdot \vec{v}/c$. However, the fields are given and since $\vec{v} = v_y \hat{e}_y + v_z \hat{e}_z$, we find

$$-\frac{\sqrt{2}}{c} E_0 v_y = -\frac{\lambda}{c} E_0 v_y.$$

From this you might be tempted to conclude that $\lambda = \sqrt{2}$, but this assumes that $v_y \neq 0$, which we don't want to assume. Thus, the only thing we can say from the above is that $(\lambda - \sqrt{2})v_y = 0$.

Now for the perpendicular components: suppose that $\hat{n} = \vec{v}/v$. We know that $E_{\|} = \vec{E} \cdot \hat{n}$, so the part of the vector parallel to the velocity is $(\vec{E} \cdot \hat{n})\hat{n}$. The remainder of the vector is the perpendicular component, so $\vec{E}_{\perp} = \vec{E} - (\vec{E} \cdot \hat{n})\hat{n}$, since the two components must add up to the total vector. Similarly, $\vec{B}_{\perp} = \vec{B} - (\vec{B} \cdot \hat{n})\hat{n}$ and the same for the primed fields. We therefore see that the transformation laws for the perpendicular components are

$$\begin{aligned} \vec{E}' - \left(\vec{E}' \cdot \hat{n}\right)\hat{n} &= \gamma(v)\left[\vec{E} - \left(\vec{E} \cdot \hat{n}\right)\hat{n} + \vec{v} \times \vec{B}\right], \\ \vec{B}' - \left(\vec{B}' \cdot \hat{n}\right)\hat{n} &= \gamma(v)\left[\vec{B} - \left(\vec{B} \cdot \hat{n}\right)\hat{n} - \frac{\vec{v}}{c^2} \times \vec{E}\right]. \end{aligned}$$

Since $\vec{B}'_{\perp} = -\lambda \vec{E}'_{\perp}/c$, the above gives us (after cancelling out a $\gamma(v)$)

$$\vec{B} - \left(\vec{B} \cdot \hat{n}\right)\hat{n} - \frac{\vec{v}}{c^2} \times \vec{E} = -\frac{\lambda}{c}\left[\vec{E} - \left(\vec{E} \cdot \hat{n}\right)\hat{n} + \vec{v} \times \vec{B}\right]$$

but since we already know from the parallel components that $\vec{B} \cdot \hat{n} = -\lambda \vec{E} \cdot \hat{n}/c$, we obtain

$$\vec{B} - \frac{\vec{v}}{c^2} \times \vec{E} = -\frac{\lambda}{c}\left[\vec{E} + \vec{v} \times \vec{B}\right].$$

Now we'll put in the explicit forms for $\vec{v}$, $\vec{E}$ and $\vec{B}$, and it's easy to see that we get (after dividing out common factors of E_0 and powers of c) the following three equations from the x-, y- and z-components:

$$\begin{aligned} \sqrt{2} + \frac{v_z}{c} &= -\frac{\sqrt{2}\lambda v_z}{c}, \\ -\sqrt{2} &= -\lambda\left(1 + \frac{\sqrt{2}v_z}{c}\right), \\ 0 &= \frac{\lambda\sqrt{2}v_y}{c}. \end{aligned}$$

Now, $\lambda > 0$, so the last of these immediately tells us that $v_y = 0$, so the boost is purely in the z-direction. (Note that this is the direction perpendicular to both fields; more on that in a bit.) Conveniently, this also solves the equation we obtained first, $(\lambda - \sqrt{2})v_y = 0$. We're therefore left with two equations for two unknowns, and now let's solve them: multiply the first through by $1+\sqrt{2}v_z/c$ and then use the second to get

$$\begin{aligned}\left(\sqrt{2}+\frac{v_z}{c}\right)\left(1+\frac{\sqrt{2}v_z}{c}\right) &= -\frac{\sqrt{2}\lambda v_z}{c}\left(1+\frac{\sqrt{2}v_z}{c}\right)\\ &= \frac{\sqrt{2}v_z}{c}\left[-\lambda\left(1+\frac{\sqrt{2}v_z}{c}\right)\right]\\ &= \frac{\sqrt{2}v_z}{c}\left(-\sqrt{2}\right)\end{aligned}$$

and this can be rearranged to give the quadratic equation $\sqrt{2}\,v_z^2+5cv_z+\sqrt{2}\,c^2=0$. This has the two roots

$$\left(\frac{-5\pm\sqrt{17}}{2\sqrt{2}}\right)c$$

but you can easily confirm with a calculator that choosing the minus gives $v_z \approx -3.23c$, which is unphysical since the speed must be less than c. Thus, the boost that makes the fields antiparallel with one another is obtained with the relative velocity

$$\begin{aligned}\vec{v} &= \left(\frac{-5+\sqrt{17}}{2\sqrt{2}}\right)c\,\hat{e}_z\\ &\approx -0.31c\,\hat{e}_z.\end{aligned}$$

(We didn't ask for λ, but if you put this velocity into the equations and solve for it, you get

$$\lambda = \frac{3+\sqrt{17}}{2\sqrt{2}}$$

which is approximately 2.52.)

Now, even though this was a specific case, we're now able to state something in general: suppose we have a nonzero electric and magnetic field in some frame $\mathcal{S}$. If the angle between them is less than

90°, then there exists a frame, boosted in the direction of $\vec{E} \times \vec{B}$, in which the fields are parallel. If the angle is greater than 90° (the case we had above), there'll be a frame boosted in this direction (the negative z-direction above) in which they're antiparallel. And if they're perpendicular, they'll be perpendicular in *all* frames, regardless of the boost.

(The proof of this theorem is not something we'll do here, but it uses the result of the very next problem. Don't you just love these segues?)

4. This is actually pretty easy if we use some of the matrix trickery I described in lecture. We said that both $\star F^{\mu\nu}$ and $F_{\mu\nu}$ can be thought of as the components of two 4×4 matrices which we'll call $\tilde{F}$ and F respectively; we gave their forms as

$$\begin{aligned} F &= \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}, \\ \tilde{F} &= \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}. \end{aligned}$$

I reminded you that if M and N are two $n \times n$ matrices, then the ij^{th} element of the matrix MN is

$$(MN)_{ij} = \sum_{k=1}^{n} M_{ik} N_{kj}$$

so any time that adjacent indices (the ks in the above) are summed over, it's matrix multiplication. This means that the matrix $\tilde{F}F$ has components

$$\begin{aligned} \left(\tilde{F}F\right)^{\mu}{}_{\lambda} &= \sum_{\nu=0}^{3} (\tilde{F})^{\mu\nu} (F)_{\nu\lambda} \\ &= \sum_{\nu=0}^{3} (\tilde{F})^{\mu\nu} (F^T)_{\lambda\nu} \end{aligned}$$

since transposition of a matrix switches the order of its indices. However, F is antisymmetric, so $F^T = -F$. But the components of these are just given by the field strength and dual field strength, and if we adopt Einstein summation convention, we see the elements of $\tilde{F}F$ are

$$\left(\tilde{F}F\right)^{\mu}{}_{\lambda} = -\star F^{\mu\nu}F_{\lambda\nu}.$$

Now, let $\lambda = \mu$ and sum over all values; this is the quantity we're after. From the above, we see this is

$$\star F^{\mu\nu}F_{\mu\nu} = -\sum_{\mu=0}^{3}\left(\tilde{F}F\right)^{\mu}{}_{\mu}.$$

But the sum over the diagonal elements of a matrix is just the trace! This means

$$\star F^{\mu\nu}F_{\mu\nu} = -\mathrm{tr}\left(\tilde{F}F\right)$$

and so all we need to do is multiply the two matrices given above and take their trace, and we've got our result.

Now, we could do this in all its horrible splendour, but because we're going to take the trace, we're actually only interested in the diagonal elements of $\tilde{F}F$, and they can be shown to all be equal to

$$\begin{aligned}\left(\tilde{F}F\right)^{0}{}_{0} = \left(\tilde{F}F\right)^{1}{}_{1} = \left(\tilde{F}F\right)^{2}{}_{2} = \left(\tilde{F}F\right)^{3}{}_{3} &= \frac{1}{c}\left(E_xB_x + E_yB_y + E_zB_z\right) \\ &= \frac{1}{c}\vec{E}\cdot\vec{B}\end{aligned}$$

and so

$$\begin{aligned}\star F^{\mu\nu}F_{\mu\nu} &= -\mathrm{tr}\left(\tilde{F}F\right) \\ &= -\left[\left(\tilde{F}F\right)^{0}{}_{0} + \left(\tilde{F}F\right)^{1}{}_{1} + \left(\tilde{F}F\right)^{2}{}_{2} + \left(\tilde{F}F\right)^{3}{}_{3}\right] \\ &= -\frac{4}{c}\vec{E}\cdot\vec{B}\end{aligned}$$

as desired.

Finally, since this quantity is constructed by contracting two 4-vector indices, with no uncontracted indices remaining, it's a Lorentz-invariant quantity.

Now that we have this, it's nearly trivial to prove the last part of the theorem quoted at the end of Problem 3: if $\vec{E}$ and $\vec{B}$ are perpendicular in some frame $\mathcal{S}$, then $\vec{E} \cdot \vec{B} = 0$ in that frame. But this is Lorentz-invariant (the constant $-4/c$ doesn't change that), so in a different inertial frame $\mathcal{S}'$, the inner product of the fields in that frame must also be zero, i.e. they're perpendicular regardless of how $\mathcal{S}$ and $\mathcal{S}'$ are related. This doesn't means $\vec{E}' = \vec{E}$ and $\vec{B}' = \vec{B}$, just that the fields transform such that there's always a right angle between them.