MP465 – Advanced Electromagnetism

Solutions to Problem Set 4

1. (a) Just to make things easier, we'll define $A = |\mathcal{E}_x|$ and $B = |\mathcal{E}_y|$, which gives the x- and y-components of $\vec{E}$ as $E_x = A\cos\theta$ and $E_y = B\cos(\theta - \delta)$. But the y-component may be rewritten $E_y = B(\cos\theta\cos\delta + \sin\theta\sin\delta)$, and thus we can solve for the sine and cosine of θ:

$$\begin{aligned} \cos\theta &= \frac{E_x}{A}, \\ \sin\theta &= \frac{E_y}{B\sin\delta} - \frac{\cos\theta\cos\delta}{\sin\delta} \\ &= \frac{E_y}{B\sin\delta} - \frac{E_x\cos\delta}{A\sin\delta}. \end{aligned}$$

Hopefully we all know that $\sin^2\theta + \cos^2\theta = 1$, and so

$$\begin{aligned} 1 &= \left(\frac{E_y}{B\sin\delta} - \frac{E_x\cos\delta}{A\sin\delta}.\right)^2 + \left(\frac{E_x}{A}\right)^2 \\ &= \frac{1}{\sin^2\delta}\left(\frac{E_y^2}{B^2} - \frac{2E_xE_y\cos\delta}{AB} + \frac{E_x^2\cos^2\delta}{A^2} + \frac{E_x^2}{A^2}\sin^2\delta\right) \end{aligned}$$

so by using $\sin^2\delta + \cos^2\delta = 1$ and multiplying through by $\sin^2\delta$, we get

$$\frac{E_x^2}{A^2} - \frac{2E_xE_y\cos\delta}{AB} + \frac{E_y^2}{B^2} = \sin^2\delta$$

as desired. Thus, in the E_xE_y-plane, the tip of $\vec{E}$ will always be on the curve this equation describes, moving around it anticlockwise as θ increases.

Now, for a bit on the curves known as *conic sections*. These curves – ellipses, parabolae and hyperbolae – have been known and studied since ancient times, and show up in a wide variety of situations. For examples, the paths of objects around a large central mass are all conic sections.

When we describe a conic section in the xy-plane, all of them have the same basic form, namely, they all consist of the points (x, y) which satisfy a quadratic equation of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ for some constants a, b, c, d, e and f (where at least one of a, b and c is nonzero). We see that if we interpret x and y in this equation as E_x and E_y, then the curve we have is of this form with $a = 1/A^2$, $b = -2\cos\delta/AB$, $c = 1/B^2$, $d = e = 0$ and $f = -\sin^2\delta$. Thus, we have some sort of conic section.

But which sort is it? If we go to the Wikipedia article "Conic Section", we see that the answer depends on the value of the *discriminant* $b^2 - 4ac$. If it's negative, we have an ellipse; zero, a parabola; and positive, a hyperbola. The values we have for our curve give $b^2 - 4ac = 4(\cos^2\delta - 1)/A^2B^2 = -4\sin^2\delta/A^2B^2$. Thus, if δ is neither 0 nor π, this is negative and so we have an ellipse. (In the case δ is one of these values, we see that the equation may be written as

$$\left(\frac{E_x}{A} \pm \frac{E_y}{B}\right)^2 = 0$$

and we have the linearly-polarised cases $E_y = \mp BE_x/A$.)

(b) Now, a possible clarification: as you probably all know, one number we can assign to an object's orbit around a central mass is its *eccentricity* e (not to be confused with Euler's number, the fundamental unit of charge or the coefficient in the conic section equation above!), with this number telling us if we have a ellipse ($0 \le e < 1$), parabola ($e = 1$) or hyperbola ($e > 1$).

This is different from the *ellipticity* that we're after; that's defined only for ellipses and not parabolae or hyperbolae. However, it's related to the eccentricity: if an ellipse has a semimajor axis of length α and a semiminor axis of length β, the ellipticity is defined as $\varepsilon = \beta/\alpha$ and the eccentricity is defined as $e = \sqrt{1 - \beta^2/\alpha^2}$, so we see that $\varepsilon = \sqrt{1 - e^2}$. Thus, if we know the eccentricity of an ellipse, we can find the ellipticity.

Luckily, the Wikipedia article has a formula for e in terms of the general conic section equation: e is the unique positive solution to the quartic equation (quadratic in e^2)

$$\Delta e^4 + [(a + c)^2 - 4\Delta]e^2 - [(a + c)^2 - 4\Delta] = 0$$

where $\Delta = ac - b^2/4$, which for us is $\sin^2\delta/A^2B^2$. If we now put in $a = 1/A^2$ and $c = 1/B^2$, we get

$$\frac{\sin^2\delta}{A^2B^2}\,e^4 + \left[\left(\frac{1}{A^2}+\frac{1}{B^2}\right)^2 - \frac{4\sin^2\delta}{A^2B^2}\right]e^2 - \left[\left(\frac{1}{A^2}+\frac{1}{B^2}\right)^2 - \frac{4\sin^2\delta}{A^2B^2}\right] \quad = \quad 0,$$

which, with a bit of cleaning up, may be rewritten as

$$(A^2B^2\sin^2\delta)e^4 + \left[A^4 + 2A^2B^2(1-2\sin^2\delta) + B^4\right]e^2$$
$$-\left[A^4 + 2A^2B^2(1-2\sin^2\delta) + B^4\right] = 0$$

Solving for e^2 gives the two solutions

$$\frac{-\left[A^4 + 2A^2B^2(1-2\sin^2\delta) + B^4\right] \pm (A^2+B^2)\sqrt{A^4 + 2A^2B^2(1-2\sin^2\delta) + B^4}}{2A^2B^2\sin^2\delta}$$

but since e is real, we need the above to be positive, so we pick the $+$ of the $\pm$:

$$e^2 \quad = \quad \frac{(A^2+B^2)\sqrt{A^4 + 2A^2B^2(1-2\sin^2\delta) + B^4} - \left[A^4 + 2A^2B^2(1-2\sin^2\delta) + B^4\right]}{2A^2B^2\sin^2\delta}.$$

But $\varepsilon^2 = 1 - e^2$, so a bit of computation gives

$$\varepsilon^2 \quad = \quad \frac{A^4 + 2A^2B^2(1-\sin^2\delta) + B^4 - (A^2+B^2)\sqrt{A^4 + 2A^2B^2(1-2\sin^2\delta) + B^4}}{2A^2B^2\sin^2\delta}.$$

Now, notice that

$$\begin{aligned}
A^4 + 2A^2B^2(1-\sin^2\delta) + B^4 \quad &= \quad \frac{1}{2}\left[2A^4 + 4A^2B^2(1-\sin^2\delta) + 2B^4\right] \\
&= \quad \frac{1}{2}\left[A^4 + 2A^2B^2 + B^4\right. \\
&\qquad \left. + A^4 + 2A^2B^2(1-2\sin^2\delta) + B^4\right] \\
&= \quad \frac{1}{2}(A^2+B^2)^2 + \frac{1}{2}\left[A^4 + 2A^2B^2(1-2\sin^2\delta) + B^4\right].
\end{aligned}$$

If we now use $1 - 2\sin^2\delta = \cos 2\delta$, then we can use all the above to rewrite the numerator of the above fraction as

$$
\begin{aligned}
& A^4 + 2A^2B^2(1-\sin^2\delta) + B^4 - (A^2+B^2)\sqrt{A^4+2A^2B^2(1-2\sin^2\delta)+B^4} \\
= \; & \frac{1}{2}\left[(A^2+B^2)^2 + (A^4+2A^2B^2\cos 2\delta + B^4)\right. \\
& \left. -2(A^2+B^2)\sqrt{A^4+2A^2B^2\cos 2\delta+B^4}\right] \\
= \; & \frac{1}{2}\left[(A^2+B^2)^2 - 2(A^2+B^2)\sqrt{A^4+2A^2B^2\cos 2\delta+B^4}\right. \\
& \left. +(\sqrt{A^4+2A^2B^2\cos 2\delta+B^4})^2\right] \\
= \; & \frac{1}{2}\left[A^2+B^2-\sqrt{A^4+2A^2B^2\cos 2\delta+B^4}\right]^2
\end{aligned}
$$

and so

$$
\varepsilon^2 = \frac{\left[A^2+B^2-\sqrt{A^4+2A^2B^2\cos 2\delta+B^4}\right]^2}{4A^2B^2\sin^2\delta}.
$$

Since ε is the ratio of two positive quantities, it must be positive as well, so the positive square root of the above is

$$
\varepsilon = \frac{A^2+B^2-\sqrt{A^4+2A^2B^2\cos 2\delta+B^4}}{2AB|\sin\delta|}.
$$

Whew. A lot of work, but this quantity really is used by experimentalists when quantifying the particular polarisation of an EM wave.

Now to take the limits mentioned. As δ gets very small we can replace the sine and cosine by the first few terms of their Taylor series expansions around zero, i.e. $\sin\delta \approx \delta$ and $\cos 2\delta \approx 1 - 2\delta^2$. We see that the square root becomes

$$
\begin{aligned}
\sqrt{A^4+2A^2B^2\cos 2\delta+B^4} &\approx \sqrt{A^4+2A^2B(1-2\delta^2)+B^4} \\
&= \sqrt{A^4+2A^2B^2+B^4-4A^2B^2\delta^2} \\
&= \sqrt{(A^2+B^2)^2\left(1-\frac{4A^2B^2}{(A^2+B^2)^2}\delta^2\right)} \\
&\approx (A^2+B^2)\left(1-\frac{2A^2B^2}{(A^2+B^2)^2}\delta^2\right)
\end{aligned}
$$

and so the numerator of ε is approximately $2A^2B^2\delta^2/(A^2+B^2)$. The denominator is approximately $2AB|\delta|$, so overall, $\varepsilon \approx AB|\delta|/(A^2+B^2)$ for small values of δ, and we see that $\varepsilon \to 0$ as δ goes to zero.

If δ is very close to π, we can define $\xi = \pi - \delta$ and so δ near π means ξ is near zero. Now, $\cos 2\delta = \cos(2\pi - 2\xi) = \cos 2\xi$ and $\sin\delta = \sin(\pi - \xi) = \sin\xi$, so we see that ε depends on ξ the exact same way as it does on δ, so in the limit $\xi \to 0$, the ellipticity goes to zero. Thus, for both instances of linear polarisation, $\varepsilon = 0$.

Now, if $A = B$, a little maths gives

$$\varepsilon = \frac{2 - \sqrt{2(1+\cos 2\delta)}}{2|\sin\delta|}$$

and since $1 + \cos 2\delta = 2\cos^2\delta$, $\varepsilon = (1 - \cos\delta)/|\sin\delta|$. Thus, we easily see that $\delta = \pm\pi/2$ gives $\epsilon = 1$ and thus describes circular polarisation.

Note that the ellipticity doesn't give *all* details about the polarisation. As we've just seen, $\varepsilon = 0$ doesn't tell us which linear polarisation (fully in phase vs. fully out of phase) we have, nor does $\varepsilon = 1$ distinguish between left-circular and right-circular polarisation. But it does immediately tell us the ratio between the maximum and minimum values of $|\vec{E}|$, which can be useful.

2. As we've mentioned (and as we saw with some actual numbers in Problem Set 3), for a lot of materials, their magnetic susceptibilities are so small that their permeabilities are extremely close to μ_0. Therefore, as long as we don't need to be too precise, $\mu \approx \mu_0$ can often be assumed.

   This has an added advantage: recall that three of the conditions at the boundary between two media are that $B_\perp$ and $\vec{H}_\parallel$ must be continuous across the boundary. In the two media, $\vec{H}_1 = \vec{B}_1/\mu_1$ and $\vec{H}_2 = \vec{B}_2/\mu_2$, so on the boundary

$$\frac{1}{\mu_1}\left(\vec{B}_1\right)_\parallel = \frac{1}{\mu_2}\left(\vec{B}_2\right)_\parallel .$$

   But if μ_1 and μ_2 are both very close to μ_0, this means that

$$\frac{1}{\mu_0}\left(\vec{B}_1\right)_\parallel \approx \frac{1}{\mu_0}\left(\vec{B}_2\right)_\parallel \;\Rightarrow\; \left(\vec{B}_1\right)_\parallel \approx \left(\vec{B}_2\right)_\parallel$$

at the boundary. In other words, *all three components* of the magnetic field are (approximately) continuous across the boundary; we don't have to worry about the magnetic intensity at all. But make sure you realise this is *only* when we can reasonably take the permeabilities to be μ_0; in cases where we can't do that, we need to treat $B_\perp$ and $\vec{H}_\parallel$ separately 'qlike we did in the example done in lecture.

But in this problem, we *are* making this approximation, so we can assume we match all components of $\vec{B}$ at the boundary. We still have to treat the continuity of $D_\perp$ and $\vec{E}_\parallel$ separately, because permittivities can take on a very wide range of values (again, as we saw in PS3).

(a) So let's do it. As in lecture, we take a planar boundary at $z = 0$ with medium 1 on the negative side and medium 2 on the positive side. In medium 1, we'll have an incident electric field $\vec{E}_I$ and a reflected field $\vec{E}_R$ and in medium 2 only a transmitted field $\vec{E}_T$ with wave vectors $\vec{k}_I$, $\vec{k}_R$ and $\vec{k}_T$ respectively. In all cases, the associated magnetic fields will be $\vec{B}_a = \vec{k}_a \times \vec{E}_a/\omega$ where $a = I$, R or T. As stated, we can assume the basic laws of optics, so we immediately know that if $\vec{k}_I$ is as given in the problem, then

$$\begin{aligned} \vec{k}_R &= \frac{n_1\omega}{c}\left(\sin\theta_R\,\hat{e}_x - \cos\theta_R\,\hat{e}_z\right), \\ \vec{k}_T &= \frac{n_2\omega}{c}\left(\sin\theta_T\,\hat{e}_x + \cos\theta_T\,\hat{e}_z\right) \end{aligned}$$

where $\theta_R = \theta_I$ and $n_2 \sin\theta_T = n_1 \sin\theta_I$.

The case we did in lecture had $\vec{E}_I$ in the same plane as the three wave vectors; now we consider the complementary case in which the incident electric field is normal to this plane, namely,

$$\begin{aligned} \vec{E}_I &= \mathrm{Re}[\tilde{\vec{E}}_0 e^{i(\vec{k}_I\cdot\vec{r}-\omega t)}] \\ &= \mathrm{Re}[\mathcal{E}_I e^{i(\vec{k}_I\cdot\vec{r}-\omega t)}]\hat{e}_y \end{aligned}$$

where we assume we know the complex number $\mathcal{E}_I$. However, we will *not* assume that the reflected and transmitted field are normal to this plane (although we'll find that as a result), only that they're perpenducular to their wave vectors. Thus, as we did in lecture, we take their amplitudes to have the form

$$\begin{aligned}\tilde{\vec{E}}_{0R} &= -\mathcal{E}_R\left(\cos\theta_R\,\hat{e}_x+\sin\theta_R\,\hat{e}_z\right)+\tilde{E}_{Ry}\hat{e}_y,\\ \tilde{\vec{E}}_{0T} &= \mathcal{E}_T\left(\cos\theta_T\,\hat{e}_x-\sin\theta_T\,\hat{e}_z\right)+\tilde{E}_{Ty}\hat{e}_y.\end{aligned}$$

We know that $D_\perp$ must be the same on either side of the boundary, and this means $\epsilon_1(\tilde{\vec{E}}_I+\tilde{\vec{E}}_R)_z=\epsilon_2(\tilde{\vec{E}}_T)_z$. Since the incident field has no z-component, this gives $-\epsilon_1\mathcal{E}_R\sin\theta_R=-\epsilon_2\mathcal{E}_T\sin\theta_T$. The continuity of the parallel components of $\vec{E}$ implies $(\tilde{\vec{E}}_I+\tilde{\vec{E}}_R)_x=(\tilde{\vec{E}}_T)_x$ and $(\tilde{\vec{E}}_I+\tilde{\vec{E}}_R)_y=(\tilde{\vec{E}}_T)_y$, which, for the fields given, means that $-\mathcal{E}_R\cos\theta_R=\mathcal{E}_T\cos\theta_T$ and $\mathcal{E}_I+\tilde{E}_{Ry}=\tilde{E}_{Ty}$ respectively.

We already have two equations for $\mathcal{E}_R$ and $\mathcal{E}_T$:

$$\begin{aligned}\epsilon_1\mathcal{E}_R\sin\theta_R-\epsilon_2\mathcal{E}_T\sin\theta_T &= 0,\\ \mathcal{E}_R\cos\theta_R+\mathcal{E}_T\cos\theta_T &= 0,\end{aligned}$$

or, in matrix form,

$$\begin{pmatrix}\epsilon_1\sin\theta_R & -\epsilon_2\sin\theta_T\\ \cos\theta_R & \cos\theta_T\end{pmatrix}\begin{pmatrix}\mathcal{E}_R\\ \mathcal{E}_T\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

This has a nonzero solution only if the determinant of the 2×2 matrix is zero: it's $\epsilon_1\sin\theta_R\cos\theta_T+\epsilon_2\sin\theta_T\cos\theta_R$. But if all angles are between 0 and $\pi/2$, both terms are strictly positive and thus this determinant cannot be zero. (The $\theta=0$ and $\theta=\pi/2$ cases must be looked at separately, but the result is the same.) Thus, the only solution is the null vector and so $\mathcal{E}_R=\mathcal{E}_T=0$ and we see that if the incident electric field is normal to the plane of the three wave vectors, so are the reflected and transmitted fields.

Now for the magnetic field: in our approximation where all permeabilities are the same, all three components must be continuous across the boundary, and their amplitudes are

$$\begin{aligned}\tilde{\vec{B}}_0=\frac{\vec{k}_I}{\omega}\times\tilde{\vec{E}}_0 &= \frac{n_1\mathcal{E}_I}{c}\left(-\cos\theta_I\,\hat{e}_x+\sin\theta_I\,\hat{e}_z\right),\\ \tilde{\vec{B}}_{0R}=\frac{\vec{k}_R}{\omega}\times\tilde{\vec{E}}_{0R} &= \frac{n_1}{c}\left[\tilde{E}_{Ry}\left(\cos\theta_R\hat{e}_x+\sin\theta_R\hat{e}_z\right)+\mathcal{E}_R\hat{e}_y\right],\\ \tilde{\vec{B}}_{0T}=\frac{\vec{k}_T}{\omega}\times\tilde{\vec{E}}_{0T} &= \frac{n_2}{c}\left[\tilde{E}_{Ty}\left(-\cos\theta_T\hat{e}_x+\sin\theta_T\hat{e}_z\right)+\mathcal{E}_T\hat{e}_y\right].\end{aligned}$$

We already know that $\mathcal{E}_R = \mathcal{E}_T = 0$ and thus the y-components automatically agree on either side of the boundary. Matching the x- and z-components yields, with a bit of cleaning up, $-n_1\mathcal{E}_I \cos\theta_I + n_1\tilde{E}_{Ry}\cos\theta_R = -n_2\tilde{E}_{Ty}\cos\theta_T$ and $n_1\mathcal{E}_I \sin\theta_I + n_1\tilde{E}_{Ry}\sin\theta_R = n_2\tilde{E}_{Ty}\sin\theta_T$. The second we already know, since $n_1\sin\theta_I = n_1\sin\theta_R = n_2\sin\theta_T$ and we recover $\mathcal{E}_I + \tilde{E}_{Ry} = \tilde{E}_{Ty}$. Thus, in matrix form, our two equations giving $\tilde{E}_{Ry}$ and $\tilde{E}_{Ty}$ are

$$\begin{pmatrix} 1 & -1 \\ n_1\cos\theta_R & n_2\cos\theta_T \end{pmatrix}\begin{pmatrix} \tilde{E}_{Ry} \\ \tilde{E}_{Ty} \end{pmatrix} = \begin{pmatrix} -1 \\ n_1\cos\theta_I \end{pmatrix}\mathcal{E}_I.$$

and solving this gives

$$\begin{aligned} \tilde{E}_{Ry} &= \left(\frac{n_1\cos\theta_I - n_2\cos\theta_T}{n_1\cos\theta_I + n_2\cos\theta_T}\right)\mathcal{E}_I, \\ \tilde{E}_{Ty} &= \left(\frac{2n_1\cos\theta_I}{n_1\cos\theta_I + n_2\cos\theta_T}\right)\mathcal{E}_I. \end{aligned}$$

Thus, with $\tilde{\vec{E}}_0 = \mathcal{E}_I\hat{e}_y$, the reflected and transmitted electric fields are

$$\begin{aligned} \vec{E}_R &= \left(\frac{n_1\cos\theta_I - n_2\cos\theta_T}{n_1\cos\theta_I + n_2\cos\theta_T}\right)\mathrm{Re}\left[\tilde{\vec{E}}_0 e^{i(\vec{k}_R\cdot\vec{r}-\omega t)}\right], \\ \vec{E}_T &= \left(\frac{2n_1\cos\theta_I}{n_1\cos\theta_I + n_2\cos\theta_T}\right)\mathrm{Re}\left[\tilde{\vec{E}}_0 e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}\right]. \end{aligned}$$

(b) In lecture, we derived forms for the two coefficients: if the three fields have the forms

$$\begin{aligned} \vec{E}_I &= \mathrm{Re}\left[\tilde{\vec{E}}_0 e^{i(\vec{k}_I\cdot\vec{r}-\omega t)}\right], \\ \vec{E}_R &= \mathrm{Re}\left[\tilde{\vec{E}}_{0R} e^{i(\vec{k}_R\cdot\vec{r}-\omega t)}\right], \\ \vec{E}_T &= \mathrm{Re}\left[\tilde{\vec{E}}_{0T} e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}\right], \end{aligned}$$

then

$$R = \frac{|\tilde{\vec{E}}_{0R}|^2}{|\tilde{\vec{E}}_0|^2}, \qquad T = \frac{n_2\mu_1\cos\theta_T}{n_1\mu_2\cos\theta_I}\,\frac{|\tilde{\vec{E}}_{0T}|^2}{|\tilde{\vec{E}}_0|^2}.$$

In (a), we found

$$\begin{aligned}
\tilde{\vec{E}}_0 &= \mathcal{E}_I \hat{e}_y, \\
\tilde{\vec{E}}_{0R} &= \left(\frac{n_1 \cos\theta_I - n_2 \cos\theta_T}{n_1 \cos\theta_I + n_2 \cos\theta_T} \right) \tilde{\vec{E}}_0, \\
\tilde{\vec{E}}_{0T} &= \left(\frac{2 n_1 \cos\theta_I}{n_1 \cos\theta_I + n_2 \cos\theta_T} \right) \tilde{\vec{E}}_0
\end{aligned}$$

so

$$\begin{aligned}
\frac{|\tilde{\vec{E}}_{0R}|^2}{|\tilde{\vec{E}}_0|^2} &= \left(\frac{n_1 \cos\theta_I - n_2 \cos\theta_T}{n_1 \cos\theta_I + n_2 \cos\theta_T} \right)^2 \\
\frac{|\tilde{\vec{E}}_{0T}|^2}{|\tilde{\vec{E}}_0|^2} &= \left(\frac{2 n_1 \cos\theta_I}{n_1 \cos\theta_I + n_2 \cos\theta_T} \right)^2 .
\end{aligned}$$

Thus, since we assume $\mu_1 \approx \mu_2 \approx \mu_0$,

$$\begin{aligned}
R &= \frac{n_1^2 \cos^2\theta_I - 2 n_1 n_2 \cos\theta_I \cos\theta_T + n_2^2 \cos^2\theta_T}{n_1^2 \cos^2\theta_I + 2 n_1 n_2 \cos\theta_I \cos\theta_T + n_2^2 \cos^2\theta_T} \\
T &= \frac{4 n_1 n_2 \cos\theta_I \cos\theta_T}{n_1^2 \cos^2\theta_I + 2 n_1 n_2 \cos\theta_I \cos\theta_T + n_2^2 \cos^2\theta_T}
\end{aligned}$$

and so

$$\begin{aligned}
R + T &= \frac{(n_1^2 \cos^2\theta_I - 2 n_1 n_2 \cos\theta_I \cos\theta_T + n_2^2 \cos^2\theta_T) + 4 n_1 n_2 \cos\theta_I \cos\theta_T}{n_1^2 \cos^2\theta_I + 2 n_1 n_2 \cos\theta_I \cos\theta_T + n_2^2 \cos^2\theta_T} \\
&= \frac{n_1^2 \cos^2\theta_I + 2 n_1 n_2 \cos\theta_I \cos\theta_T + n_2^2 \cos^2\theta_T}{n_1^2 \cos^2\theta_I + 2 n_1 n_2 \cos\theta_I \cos\theta_T + n_2^2 \cos^2\theta_T}
\end{aligned}$$

which is, of course, 1.