MP465 – Advanced Electromagnetism

Solutions to Problem Set 2

1. We know, of course, that $\vec{E}_1 = -\vec{\nabla}\Phi_1$, or if we use Cartesian coordinates, $E_{1i} = -\partial\Phi_1/\partial x_i$. The first thing we need is the derivative of the radial coordinate r with respect to x_i:

$$\begin{aligned} \frac{\partial}{\partial x_i} r &= \frac{\partial}{\partial x_i}\sqrt{\vec{r}\cdot\vec{r}} \\ &= \frac{1}{2\sqrt{\vec{r}\cdot\vec{r}}}\frac{\partial}{\partial x_i}\left(\vec{r}\cdot\vec{r}\right) \\ &= \frac{1}{2r}\left(\frac{\partial}{\partial x_i}\sum_j x_j^2\right) \\ &= \frac{1}{2r}\left(2\sum_j x_j \frac{\partial x_j}{\partial x_i}\right). \end{aligned}$$

The derivative of x_j with respect to x_i is 1 if $i = j$ and 0 if $i \neq j$; in other words, the Kronecker delta-symbol δ_{ij}. Thus,

$$\begin{aligned} \frac{\partial}{\partial x_i} r &= \frac{1}{r}\left(\sum_j x_j \delta_{ij}\right) \\ &= \frac{x_i}{r}. \end{aligned}$$

We also need the derivative of $\vec{p}\cdot\vec{r}$, but this is easy now that we have $\partial x_j/\partial x_i = \delta_{ij}$:

$$\begin{aligned} \frac{\partial}{\partial x_i}\vec{p}\cdot\vec{r} &= \frac{\partial}{\partial x_i}\sum_j p_j x_j \\ &= \sum_j p_j \frac{\partial x_j}{\partial x_i} \\ &= p_i. \end{aligned}$$

So with all this, we see

$$
\begin{aligned}
\frac{\partial}{\partial x_i}\left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right) &= \frac{\left(\frac{\partial}{\partial x_i}\,\vec{p}\cdot\vec{r}\right)(r^3) - (\vec{p}\cdot\vec{r})\left(3r^2\frac{\partial}{\partial x_i}\,r\right)}{r^6} \\
&= \frac{p_i r^3 - 3r(\vec{p}\cdot\vec{r})x_i}{r^6}
\end{aligned}
$$

and so

$$
\begin{aligned}
\left(\vec{E}_1\right)_i &= -\frac{\partial \Phi_1}{\partial x_i} \\
&= -\frac{1}{4\pi\epsilon_0}\left(\frac{p_i r^3 - 3r(\vec{p}\cdot\vec{r})x_i}{r^6}\right)
\end{aligned}
$$

and this, after cleaning this up a little and putting back in the unit vectors, gives the desired answer:

$$
\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\,\frac{3\,(\vec{p}\cdot\vec{r})\,\vec{r} - r^2\vec{p}}{r^5}.
$$

2. (a) The monopole moment is simply the total electric charge, i.e.

$$
\begin{aligned}
q &= \int \rho\,\mathrm{d}^3\vec{r} \\
&= \int \sigma_0 \sin(2\phi)\delta(r-R)\,r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \sigma_0 R^2\left(\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\right)\left(\int_0^{2\pi}\sin(2\phi)\,\mathrm{d}\phi\right)
\end{aligned}
$$

where we've done the r-integral. The θ-intgral is 2, and using the identity $\sin(2\phi) = 2\sin\phi\cos\phi$, this becomes

$$
q = 4\sigma_0 R^2 \int_0^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi
$$

which vanishes when we remember that, if m and n are nonnegative integers, the integral

$$
\int_0^{2\pi}\sin^m\phi\cos^n\phi\,\mathrm{d}\phi
$$

is nonzero only if *both* m and n are even. Thus, this shell has no electric monopole moment.

The dipole moment is $\vec{p} = \int \rho \vec{r} \, \mathrm{d}^3\vec{r}$, so let's look at it component-by-component:

$$\begin{aligned} p_x &= \int \rho x \, \mathrm{d}^3\vec{r} \\ &= \int \left(\sigma_0 \sin(2\phi)\delta(r-R)\right)\left(r\sin\theta\cos\phi\right) r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= \sigma_0 R^3 \left(\int_0^{\pi} \sin^2\theta \, \mathrm{d}\theta\right)\left(\int_0^{2\pi} \sin(2\phi)\cos\phi \, \mathrm{d}\phi\right). \end{aligned}$$

Since $\sin(2\phi)\cos\phi = 2\sin\phi\cos^2\phi$, we see that the ϕ-integral will vanish because of the integral identity we quoted above, and so $p_x = 0$.

p_y will be a similar integral, except with $x = r\sin\theta\cos\phi$ replaced by $y = r\sin\theta\sin\phi$; this means that the integrand of the ϕ-integral will be $\sin(2\phi)\sin\phi = 2\sin^2\phi\cos\phi$, and the integral of this will also vanish, giving $p_y = 0$. And finally, the z-component will use $z = \cos\theta$, so the ϕ integrand will be $\sin(2\phi) = 2\sin\phi\cos\phi$ and so will also integrate to zero. Thus, all three components vanish and so $\vec{p} = \vec{0}$: the shell has no electric dipole moment.

(b) For the quadrupole moment, because Q is a symmetric and traceless matrix, we need only compute five quantities. Let's look at Q_{xy}, Q_{xz}, Q_{yz}, Q_{xx} and Q_{yy}. As you can probably gather, we'll concentrate on how the integrands involved depend on ϕ, because for most of these we'll see quickly that they vanish.

Recall the general formula:

$$Q_{ij} = \frac{1}{2}\int \rho\left(3x_i x_j - |\vec{r}|^2\delta_{ij}\right) \mathrm{d}^3\vec{r}.$$

So to get Q_{xx}, we pick $i = j = 1$ and so the quantity in the brackets will be

$$\begin{aligned} 3x_1^2 - |\vec{r}|^2\delta_{11} &= 2x^2 - y^2 - z^2 \\ &= r^2\left(2\sin^2\theta\cos^2\phi - \sin^2\theta\sin^2\phi - \cos^2\theta\right). \end{aligned}$$

This gets multiplied by ρ and so will pick up a factor of $\sin(2\phi) = 2\sin\phi\cos\phi$. Thus, looking only at the ϕ-dependence, the first term will have one sine and three cosines, the second three sines and one cosine and the last one sine and one cosine. Our integral identity thus says each will integrate to zero, and thus with not too much effort we see that $Q_{xx} = 0$. Q_{yy} will have $i = j = 2$, or

$$\begin{aligned} 3x_2^2 - |\vec{r}|^2\delta_{22} &= 2y^2 - x^2 - z^2 \\ &= r^2\left(2\sin^2\theta\sin^2\phi - \sin^2\theta\cos^2\phi - \cos^2\theta\right). \end{aligned}$$

and a virtually-identical argument as above tells us $Q_{yy} = 0$. And since $Q_{xx} + Q_{yy} + Q_{zz} = 0$, this gives $Q_{zz} = 0$.

Now for the off-diagonal components: Q_{xz} has

$$\begin{aligned} 3x_1x_3 - |\vec{r}|^2\delta_{13} &= 3xz \\ &= 3r^2\sin\theta\cos\theta\cos\phi \end{aligned}$$

so when multiplied by $\sin(2\phi)$ it will have one sine and two cosines, and thus integrates to zero (and also gives $Q_{zx} = 0$ as well). $Q_{yz} = Q_{zy}$ involves

$$\begin{aligned} 3x_2x_3 - |\vec{r}|^2\delta_{23} &= 3yz \\ &= 3r^2\sin\theta\cos\theta\sin\phi \end{aligned}$$

and thus the integral over ϕ will be of $\sin^2\phi\cos\phi$ and will give zero.

$Q_{xy} = Q_{yx}$ are all that's left, and we have

$$\begin{aligned} 3x_1x_2 - |\vec{r}|^2\delta_{12} &= 3xy \\ &= 3r^2\sin^2\theta\sin\phi\cos\phi \end{aligned}$$

and so when multiplied by $\sin(2\phi)$ will give a $\sin^2\phi\cos^2\phi$, which does *not* integrate to zero. Now that we see this, let's do the computation in full:

$$\begin{aligned} Q_{xy} &= \frac{1}{2}\int \rho(3xy)\,\mathrm{d}^3\vec{r} \\ &= \frac{1}{2}\int (\sigma_0\sin(2\phi)\delta(r-R))\,(3r^2\sin^2\theta\sin\phi\cos\phi)\,r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= 3\sigma_0R^4\left(\int_0^{\pi}\sin^3\theta\,\mathrm{d}\theta\right)\left(\int_0^{2\pi}\sin^2\phi\cos^2\phi\,\mathrm{d}\phi\right) \end{aligned}$$

The θ-integral is easily done with the substitution $\mu = \cos\theta$, and I'll leave it to you to show you get $4/3$. I will, however, show the ϕ-integral computation:

$$\begin{aligned}
\int_0^{2\pi} \sin^2\phi\cos^2\phi\,\mathrm{d}\phi &= \frac{1}{4}\int_0^{2\pi} (2\sin\phi\cos\phi)^2\,\mathrm{d}\phi \\
&= \frac{1}{4}\int_0^{2\pi} \sin^2(2\phi)\,\mathrm{d}\phi \\
&= \frac{1}{4}\int_0^{2\pi} \left[\frac{1}{2} - \frac{1}{2}\cos(4\phi)\right]\,\mathrm{d}\phi \\
&= \frac{1}{4}\left[\frac{\phi}{2} - \frac{1}{8}\sin(4\phi)\right|_0^{2\pi}
\end{aligned}$$

which gives $\pi/4$ and thus $Q_{xy} = Q_{yx} = \pi\sigma_0 R^4$ are the only nonzero components of the electric quadrupole moment matrix.

(c) Written as a matrix, we see from (b) that

$$Q = \pi\sigma_0 R^4 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The quadrupole contribution is written in terms of $\vec{r}^T \cdot Q \cdot \vec{r}$, where we here interpret $\vec{r}$ as a Cartesian column vector, i.e. $\vec{r}^T = (x\ y\ z)$. Thus,

$$\begin{aligned}
\vec{r}^T \cdot Q \cdot \vec{r} &= \pi\sigma_0 R^4 \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= 2\pi\sigma_0 R^4 xy \\
&= 2\pi\sigma_0 R^4 r^2 \sin^2\theta\sin\phi\cos\phi
\end{aligned}$$

and so

$$\begin{aligned}
\Phi_2(r,\theta,\phi) &= \frac{1}{4\pi\epsilon_0}\frac{\vec{r}^T\cdot Q\cdot\vec{r}}{r^5} \\
&= \frac{\sigma_0 R^4}{2\epsilon_0}\frac{\sin^2\theta\sin\phi\cos\phi}{r^3}
\end{aligned}$$

is the electric quadrupole contribution to the scalar potential.

3. We start, naturally enough, with the formula giving the magnetic field:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,\mathrm{d}^3\vec{r}'.$$

On the z-axis, $\vec{r} = z\hat{e}_z$, so

$$\begin{aligned}\vec{r}-\vec{r}' &= z\hat{e}_z - r'\hat{e}_{r'} \\ &= (z - r'\cos\theta')\hat{e}_z - r'\sin\theta'\cos\phi'\,\hat{e}_x - r'\sin\theta'\sin\phi'\,\hat{e}_y\end{aligned}$$

and from this you can quickly show that

$$|\vec{r}-\vec{r}'| = \sqrt{(r')^2 - 2zr'\cos\theta' + z^2}$$

and, using $\hat{e}_{\phi'} = -\sin\phi'\,\hat{e}_x + \cos\phi'\,\hat{e}_y$,

$$\hat{e}_{\phi'}\times(\vec{r}-\vec{r}') = (z - r'\cos\theta')(\cos\phi'\,\hat{e}_x + \sin\phi'\,\hat{e}_y) + r'\sin\theta'\,\hat{e}_z.$$

Thus, on the z-axis,

$$\begin{aligned}\vec{B} &= \frac{\mu_0}{4\pi}\int \frac{K\sin\theta'\delta(r'-R)\hat{e}_{\phi'}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,\mathrm{d}^3\vec{r}' \\ &= \frac{\mu_0 K}{4\pi}\int \frac{(z - r'\cos\theta')(\cos\phi'\,\hat{e}_x + \sin\phi'\,\hat{e}_y) + r'\sin\theta'\,\hat{e}_z}{[(r')^2 - 2zr'\cos\theta' + z^2]^{3/2}}\sin\theta'\,\delta(r'-R) \\ &\quad\times(r')^2\sin\theta'\mathrm{d}r'\mathrm{d}\theta'\mathrm{d}\phi'.\end{aligned}$$

This immediately simplifies in two ways: first, doing the r'-integral replaces r' by R because of the delta-function, and second, because of the $\sin\phi'$ and $\cos\phi'$ in the x- and y-components, they vanish under integration leaving only the z-component, and then the ϕ'-integral gives a factor of 2π, with the upshot being that we only have the θ' integral left:

$$\vec{B} = \frac{\mu_0 K R^3\hat{e}_z}{2}\int_0^\pi \frac{\sin^3\theta'}{(R^2 - 2zR\cos\theta' + z^2)^{3/2}}\,\mathrm{d}\theta'.$$

Now we're in a position to use the integrals given. If we make the suggested variable change, then

$$
\begin{aligned}
\vec{B} &= \frac{\mu_0 K R^3 \hat{e}_z}{2} \int_0^\pi \frac{1-\cos^2\theta'}{(R^2 - 2zR\cos\theta' + z^2)^{3/2}} (\sin\theta' \, \mathrm{d}\theta') \\
&= \frac{\mu_0 K R^3 \hat{e}_z}{2} \int_1^{-1} \frac{1-\mu^2}{(R^2 - 2zR\mu + z^2)^{3/2}} (-\mathrm{d}\mu) \\
&= \frac{\mu_0 K R^3 \hat{e}_z}{2} \int_{-1}^{1} \frac{1-\mu^2}{(R^2 - 2zR\mu + z^2)^{3/2}} \, \mathrm{d}\mu \\
&= \frac{\mu_0 K R^3 \hat{e}_z}{2} \left(\int_{-1}^{1} \frac{1}{(\alpha - \beta\mu)^{3/2}} \, \mathrm{d}\mu - \int_{-1}^{1} \frac{\mu^2}{(\alpha - \beta\mu)^{3/2}} \, \mathrm{d}\mu \right)
\end{aligned}
$$

where $\alpha = z^2 + R^2$ and $\beta = 2zR$.

The first integral gives

$$
\begin{aligned}
\int_{-1}^{1} \frac{1}{(\alpha - \beta\mu)^{3/2}} \, \mathrm{d}\mu &= \left[\frac{2}{\beta\sqrt{\alpha - \beta\mu}} \right|_{-1}^{1} \\
&= \frac{2}{\beta\sqrt{\alpha - \beta}} - \frac{2}{\beta\sqrt{\alpha + \beta}} \\
&= \frac{1}{zR} \left(\frac{1}{\sqrt{z^2 - 2zR + R^2}} - \frac{1}{\sqrt{z^2 + 2zr + R^2}} \right) \\
&= \frac{1}{zR} \left(\frac{1}{|z-R|} - \frac{1}{|z+R|} \right).
\end{aligned}
$$

Using the same ideas, you find that the second integral is

$$
\begin{aligned}
\int_{-1}^{1} \frac{\mu^2}{(\alpha - \beta\mu)^{3/2}} \, \mathrm{d}\mu = \frac{1}{4z^3R^3} \Bigg[& (z^2+R^2)^2 \left(\frac{1}{|z-R|} - \frac{1}{|z+R|} \right) \\
& + 2(z^2+R^2)\left(|z-R| - |z+R|\right) - \frac{1}{3}\left(|z-R|^3 - |z+R|^3\right) \Bigg]
\end{aligned}
$$

These results hold for both this problem and Problem 4, because the only assumption that we've made is that we're looking at $\vec{B}$ on the z-axis. However, now we specify that we're *inside* the sphere. This is equivalent to the condition that $-R < z < R$ and allows us to deal with all the absolute values: since $z + R$ will always be positive, then $|z+R| = z+R$, and since $z-R$ is negative in this region, $|z-R| = R-z$.

Thus, the first integral becomes

$$\int_{-1}^{1} \frac{1}{(\alpha-\beta\mu)^{3/2}}\,\mathrm{d}\mu = \frac{1}{zR}\left(\frac{1}{R-z}-\frac{1}{R+z}\right) = \frac{2}{R(R^2-z^2)}$$

and the second can, with some algebra, be shown to be

$$\int_{-1}^{1} \frac{\mu^2}{(\alpha-\beta\mu)^{3/2}}\,\mathrm{d}\mu = \frac{2(2z^2+R^2)}{3R^3(R^2-z^2)}$$

and so the magnetic field becomes

$$\vec{B} = \frac{\mu_0 K R^3 \hat{e}_z}{2}\left(\frac{2}{R(R^2-z^2)} - \frac{2(2z^2+R^2)}{3R^3(R^2-z^2)}\right) = \frac{\mu_0 K R^3 \hat{e}_z}{2}\left(\frac{4}{3R^3}\right) = \frac{2\mu_0 K}{3}\,\hat{e}_z$$

which is indeed constant. Proving it's constant *everywhere* inside the shell, rather than just on the z-axis, requires a different approach than the one we've done here, but is doable.

4. (a) If we're now *outside* the sphere, then we have to change the region of z-values from $-R < z < R$ to $z > R$ and $z < -R$. What are $|z-R|$ and $|z+R|$ in this region?

   When $z > R$, both $z-R$ and $z+R$ are positive, so $|z-R| = z-R$ and $|z+R| = z+R$. We then see that

$$\int_{-1}^{1} \frac{1}{(\alpha-\beta\mu)^{3/2}}\,\mathrm{d}\mu = \frac{1}{zR}\left(\frac{1}{|z-R|}-\frac{1}{|z+R|}\right) = \frac{1}{zR}\left(\frac{1}{z-R}-\frac{1}{z+R}\right) = \frac{2}{z(z^2-R^2)}.$$

   However, if $z < -R$, then both $z-R$ and $z+R$ are negative, so $|z-R| = -(z-R)$ and $|z+R| = -(z+R)$. But putting this into the above integral only changes the sign, i.e. it gives $-2/z(z^2 - R^2)$. But since $z = |z|$ in the first case and $z = -|z|$ in the second, they can be combined into a single form that's good everywhere outside the sphere:

$$\int_{-1}^{1} \frac{1}{(\alpha - \beta\mu)^{3/2}}\,\mathrm{d}\mu \quad = \quad \frac{2}{|z|(z^2 - R^2)}.$$

A similar argument and much the same algebra as we did beforehand will give

$$\int_{-1}^{1} \frac{\mu^2}{(\alpha - \beta\mu)^{3/2}}\,\mathrm{d}\mu \quad = \quad \frac{2(2R^2 + z^2)}{3|z|^3(z^2 - R^2)}$$

and so the magnetic field outside the sphere is

$$\begin{aligned}
\vec{B} &= \frac{\mu_0 K R^3 \hat{e}_z}{2}\left(\frac{2}{|z|(z^2 - R^2)} - \frac{2(2R^2 + z^2)}{3|z|^3(z^2 - R^2)}\right) \\
&= \frac{\mu_0 K R^3 \hat{e}_z}{2}\left(\frac{4}{3|z|^3}\right) \\
&= \frac{2\mu_0 K R^3}{3|z|^3}\,\hat{e}_z
\end{aligned}$$

(where at a couple of points we used $z^2 = |z|^2$).

(b) Now to show that this agrees with the dipole field formula

$$\vec{B} \quad = \quad \frac{\mu_0}{4\pi}\,\frac{3(\vec{m}\cdot\vec{r})\vec{r} - r^2\vec{m}}{r^5}.$$

when we're on the z-axis, but to do so we obviously need the shell's dipole moment. But this isn't too hard:

$$\begin{aligned}
\vec{m} &= \frac{1}{2}\int \vec{r} \times \vec{J}(\vec{r})\,\mathrm{d}^3\vec{r} \\
&= \frac{1}{2}\int r\hat{e}_r \times K\sin\theta\,\delta(r - R)\hat{e}_\phi\, r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= -\frac{KR^3}{2}\int \hat{e}_\theta \sin^2\theta\,\mathrm{d}\theta\,\mathrm{d}\phi.
\end{aligned}$$

Now, $\hat{e}_\theta = \cos\theta\cos\phi\,\hat{e}_x + \cos\theta\sin\phi\,\hat{e}_y - \sin\theta\,\hat{e}_z$, and we notice that doing the ϕ integral gets rid of the x- and y-components, so

$$
\begin{aligned}
\vec{m} &= -\frac{KR^3}{2}\int(-\sin\theta\,\hat{e}_z)\sin^2\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \pi KR^3\hat{e}_z\int_0^\pi \sin^3\theta\,\mathrm{d}\theta \\
&= \frac{4\pi KR^3}{3}\,\hat{e}_z.
\end{aligned}
$$

Unsurprisingly, the dipole is in the z-direction, so we see $\vec{m}\cdot\vec{r} = mz$. But on the z-axis, $\vec{r} = z\hat{e}_z$ and $r = |\vec{r}| = |z|$, so

$$
\begin{aligned}
\vec{B} &= \frac{\mu_0}{4\pi}\,\frac{3(\vec{m}\cdot\vec{r})\vec{r} - r^2\vec{m}}{r^5} \\
&= \frac{\mu_0}{4\pi}\,\frac{3(mz)(z\hat{e}_z) - |z|^2 m\hat{e}_z}{|z|^5} \\
&= \frac{\mu_0 m}{2\pi|z|^3}\,\hat{e}_z
\end{aligned}
$$

since $z^2 = |z|^2$. And finally, because $m = 4\pi KR^3/3$, we see this gives

$$
\vec{B} = \frac{2\mu_0 KR^3}{3|z|^3}\,\hat{e}_z
$$

which agrees with (a). Again, showing the dipole formula gives the correct field everywhere outside the shell may be done, but we won't do it here. (For the curious, Jackson does it in his book. I don't have the exact page number handy, but it's in the chapter on magnetostatics, if I remember correctly.)