MP465 – Advanced Electromagnetism

Tutorial 9 (21 April 2020)

Green's Function for the d'Alembertian

When we considered purely time-independent system, we saw that Maxwell's equations implied the following equations that the potentials had to satisfy:

$$\nabla^2\Phi = -\frac{\rho(\vec{r})}{\epsilon_0}, \qquad \nabla^2\vec{A} = -\mu_0\vec{J}(\vec{r}).$$

These are linear inhomogeneous partial differential equations (PDEs), and we can solve them using a straightforward extension of what we learned in our ordinary differential equations module: if we have a linear operator $L_{\vec{r}}$ and a function $g(\vec{r})$ and wish to find a function $y(\vec{r})$ which solves the PDE $L_{\vec{r}}\,y = g$ in a region $\mathcal{V}$, we can do so by finding a Green's function $G(\vec{r};\vec{r}\,')$ which satisfies Green's equation, i.e.

$$L_{\vec{r}}\,G(\vec{r};\vec{r}\,') \;=\; \delta^{(3)}(\vec{r}-\vec{r}\,').$$

Once we have this Green's function, then the general solution to $L_{\vec{r}}\,y = g$ is

$$y(\vec{r}) \;=\; y_0(\vec{r}) + \int_{\mathcal{V}} G(\vec{r};\vec{r}\,')g(\vec{r}\,')\,\mathrm{d}^3\vec{r}\,'.$$

where y_0 is a solution to the homogenous PDE $L_{\vec{r}}\,y = 0$.

For the static cases we started with, the Laplacian ∇^2 is the operator appearing, and we showed that a Green's function for this is the famous

$$G(\vec{r};\vec{r}\,') \;=\; -\frac{1}{4\pi}\,\frac{1}{|\vec{r}-\vec{r}\,'|}$$

and it's this that gives us all the familiar expressions for Φ and $\vec{A}$ (and, by extension, $\vec{E}$ and $\vec{B}$).

But before we extend this to the time-dependent case, let's do a quick proof. Suppose $L_{\vec{r}}$ is a *translation-invariant* operator. What this means is that if $\vec{a}$ is a constant vector, and we replace $\vec{r}$ by $\vec{r}+\vec{a}$ everywhere in $L_{\vec{r}}$, then the operator is translation-invariant if it does not change form, namely,

$L_{\vec{r}+\vec{a}} = L_{\vec{r}}$. A concrete example of such an operator in an ODE context would be $L_x = \mathrm{d}/\mathrm{d}x$; if x is replaced by $x + a$, then

$$\begin{aligned} L_{x+a} &= \frac{\mathrm{d}}{\mathrm{d}(x+a)} \\ &= \frac{\mathrm{d}x}{\mathrm{d}(x+a)} \frac{\mathrm{d}}{\mathrm{d}x} \\ &= \frac{\mathrm{d}[(x+a)-a]}{\mathrm{d}(x+a)} \frac{\mathrm{d}}{\mathrm{d}x} \\ &= \frac{\mathrm{d}}{\mathrm{d}x} \\ &= L_x. \end{aligned}$$

(As a counterexample, note that the operator $L_x = x(\mathrm{d}/\mathrm{d}x)$ is not translation-invariant, because $L_{x+a} = L_x + a(\mathrm{d}/\mathrm{d}x) \neq L_x$.)

The Green's functions of translation-invariant operators have a special property which we'll now determine. The defining relation for G is Green's equation

$$L_{\vec{r}}\, G(\vec{r};\vec{r}\,') = \delta^{(3)}(\vec{r} - \vec{r}\,'),$$

so let's shift the two position vectors by $\vec{a}$ and see what we get:

$$\begin{aligned} L_{\vec{r}+\vec{a}}\, G(\vec{r}+\vec{a};\vec{r}\,'+\vec{a}) &= \delta^{(3)}\left((\vec{r}+\vec{a}) - (\vec{r}\,'+\vec{a})\right) \\ &= \delta^{(3)}(\vec{r} - \vec{r}\,'). \end{aligned}$$

If the operator is translation-invariant, then this becomes

$$L_{\vec{r}}\, G(\vec{r}+\vec{a};\vec{r}\,'+\vec{a}) = \delta^{(3)}(\vec{r} - \vec{r}\,')$$

and so $G(\vec{r}+\vec{a};\vec{r}\,'+\vec{a})$ satisfies the same equation as $G(\vec{r};\vec{r}\,')$ and so they can be taken to be the same. But now suppose we pick $\vec{a} = -\vec{r}\,'$; this is legit because from the point of view of the operator, the primed position vector is treated as a constant. With this choice, we get the result

$$G(\vec{r}+\vec{a};\vec{r}\,'+\vec{a}) = G(\vec{r}-\vec{r}\,';\vec{0}).$$

In other words, the Green's function for any translation-invariant operator depends not on $\vec{r}$ and $\vec{r}\,'$ separately, but only on the *difference* $\vec{r} - \vec{r}\,'$. And we've seen this already: the Laplacian can easily be shown to be translation-invariant, and we see its Green's function depends only on $\vec{r} - \vec{r}'$.

So the upshot is that for translation-invariant operators, the Green's function may be found by looking for a function $\tilde{G}(\vec{r})$ satisfying

$$L_{\vec{r}}\, \tilde{G}(\vec{r}) \;=\; \delta^{(3)}(\vec{r})$$

and then getting the full Green's function from $G(\vec{r}; \vec{r}') = \tilde{G}(\vec{r} - \vec{r}')$.

Okay, back to physics. In the next lecture, we'll see that if we *don't* make an assumption of time-independence, then (with a specific choice of gauge which I'll elaborate on in the lecture) Maxwell's equations for the time-dependent potentials $\Phi(t, \vec{r})$ and $\vec{A}(t, \vec{r})$ are

$$\Box\, \Phi = -\frac{\rho(t, \vec{r})}{\epsilon_0}, \qquad \Box\, \vec{A} = -\mu_0 \vec{J}(t, \vec{r})$$

and we see it's not the Laplacian that appears, but rather the *d'Alembertian* or *wave operator*

$$\begin{aligned} \Box \;&=\; \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \\ &=\; \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}, \end{aligned}$$

where c is the speed of light (or, for a linear medium, the speed of light in the medium). Despite this change, we want to try to solve them in the same way as we did in the satic case, namely, find the Green's function and express the particular solutions as integrals over the sources.

So what we need is a Green's function for the d'Alembertian, namely, a function $G(t, \vec{r}; t', \vec{r}')$ satisfying

$$\Box_x\, G(x; x') \;=\; \delta^{(4)}(x - x')$$

where x represents the full *four*-dimensional spacetime vector $x = (t, \vec{r})$. To be somewhat more concrete, if we separate out the time and space dependence, the above is equivalent to

$$\Box_{t,\vec{r}}\, G(t, \vec{r}; t', \vec{r}') \;=\; \delta(t - t')\delta^{(3)}(\vec{r} - \vec{r}')$$

If we can find such a function, then we see that the scalar potential will be

$$\Phi(t,\vec{r}) = \Phi_0(t,\vec{r}) - \frac{1}{\epsilon_0}\int G(t,\vec{r};t',\vec{r}')\rho(t',\vec{r}')\,\mathrm{d}t'\,\mathrm{d}\vec{r}'$$

with $\Box\,\Phi_0 = 0$, and a similar expression for the vector potential. But it's easy to show that the d'Alembertian is translation-invariant, and so the problem becomes finding a solution to

$$\Box\,\tilde{G}(t,\vec{r}) = \delta(t)\delta^{(3)}(\vec{r})$$

and then getting the Green's function from this via $G(t,\vec{r};t',\vec{r}') = \tilde{G}(t-t',\vec{r}-\vec{r}')$.

Now, an actual derivation of the appropriate Green's fucntion may be done, but it requires some knowledge of complex analysis. I'd say it's highly likely that most, if not all, of you have this knowledge, but since a module in complex analysis isn't required, I'm not going to do it this way. Instead, I'm going to go the unsatisfying – but correct – route of telling you what $\tilde{G}$ is and then showing it satisfies the correct equation.

So here we go: consider the function

$$\tilde{G}(t,\vec{r}) = -\frac{1}{4\pi}\frac{\delta\left(t-\frac{r}{c}\right)}{r}$$

where $r = |\vec{r}|$. We want to compute the d'Alembertian of this, part of which will require the Laplacian. Now, $\tilde{G}$ is the product of $-1/4\pi r$ and $\delta(t-r/c)$, and so we can use the handy and easy-to-prove identity

$$\nabla^2(fg) = (\nabla^2 f)g + 2\vec{\nabla}f\cdot\vec{\nabla}g + f(\nabla^2 g)$$

to get

$$\nabla^2\tilde{G} = \left[\nabla^2\left(-\frac{1}{4\pi r}\right)\right]\delta\left(t-\frac{r}{c}\right) + 2\vec{\nabla}\left(-\frac{1}{4\pi r}\right)\cdot\vec{\nabla}\delta\left(t-\frac{r}{c}\right) - \frac{1}{4\pi r}\left[\nabla^2\delta\left(t-\frac{r}{c}\right)\right]$$

But we already know that the Laplacian of $-1/4\pi r$ is $\delta^{(3)}(\vec{r})$, so the first term is known. The second and third terms can be computed using the chain rule and $\vec{\nabla}r = \hat{e}_r$:

$$\begin{aligned} 2\vec{\nabla}\left(-\frac{1}{4\pi r}\right)\cdot\vec{\nabla}\delta\left(t-\frac{r}{c}\right) &= 2\left(\frac{\hat{e}_r}{4\pi r^2}\right)\cdot\left[-\frac{\hat{e}_r}{c}\delta'\left(t-\frac{r}{c}\right)\right] \\ &= -\frac{1}{2\pi c r^2}\,\delta'\left(t-\frac{r}{c}\right) \end{aligned}$$

and

$$
\begin{aligned}
-\frac{1}{4\pi r}\left[\nabla^2\delta\left(t-\frac{r}{c}\right)\right] &= -\frac{1}{4\pi r}\vec{\nabla}\cdot\left[-\frac{\hat{e}_r}{c}\delta'\left(t-\frac{r}{c}\right)\right] \\
&= -\frac{1}{4\pi r^3}\frac{\partial}{\partial r}\left[-\frac{r^2}{c}\delta'\left(t-\frac{r}{c}\right)\right] \\
&= -\frac{1}{4\pi r^3}\left[-\frac{2r}{c}\delta'\left(t-\frac{r}{c}\right)+\frac{r^2}{c^2}\delta''\left(t-\frac{r}{c}\right)\right] \\
&= \frac{1}{2\pi c r^2}\,\delta'\left(t-\frac{r}{c}\right)-\frac{1}{4\pi c^2 r}\,\delta''\left(t-\frac{r}{c}\right)
\end{aligned}
$$

where the primes on the delta-functions denote derivatives. (These aren't strictly "functions", but like the delta-function, may be thought of as limits of continuous functions.)

So, if we put everything together, we obtain the Laplacian we need:

$$
\nabla^2\tilde{G} = \delta\left(t-\frac{r}{c}\right)\delta^{(3)}(\vec{r})-\frac{1}{4\pi c^2 r}\,\delta''\left(t-\frac{r}{c}\right).
$$

However, the first term is nonzero only if $r=0$, so we can replace $t-r/c$ by t without changing anything, giving

$$
\nabla^2\tilde{G} = \delta(t)\delta^{(3)}(\vec{r})-\frac{1}{4\pi c^2 r}\,\delta''\left(t-\frac{r}{c}\right).
$$

Notice that the first term is precisely the 4-dimensional delta-function we want in the equation that $\tilde{G}$ must satisfy. However, we haven't yet computed the time-derivatives in the d'Alembertian, but this is easy because of the simple t-dependence in the delta-function:

$$
\begin{aligned}
\frac{\partial^2\tilde{G}}{\partial t^2} &= -\frac{1}{4\pi r}\frac{\partial^2}{\partial t^2}\delta\left(t-\frac{r}{c}\right) \\
&= -\frac{1}{4\pi r}\delta''\left(t-\frac{r}{c}\right)
\end{aligned}
$$

and we see that when we multiply this by $1/c^2$, it's exactly the second term in the Laplacian, and therefore will cancel it out in the d'Alembertian:

$$
\begin{aligned}
\Box\,\tilde{G} &= \nabla^2\tilde{G}-\frac{1}{c^2}\frac{\partial^2\tilde{G}}{\partial t^2} \\
&= \delta(t)\delta^{(3)}(\vec{r})
\end{aligned}
$$

and this is exactly the equation that $\tilde{G}$ must satisfy.

So to get the Green's function, we replace t by $t - t'$ and $\vec{r}$ by $\vec{r} - \vec{r}\,'$. But be careful; $\tilde{G}$ is written in terms of $r = |\vec{r}|$, so r is not replaced by $r - r'$ but rather by $|\vec{r} - \vec{r}\,'|$. Therefore, we get what we're after, the Green's function for the d'Alembertian operator:

$$G(t, \vec{r}; t', \vec{r}\,') = -\frac{1}{4\pi} \frac{\delta\left(t - t' - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r} - \vec{r}\,'|}.$$

But there's a bit of an issue here. Recall that Green's functions aren't unique; in this case, we could add any function whose d'Alembertian vanishes to the result we just got and obtain another solution to Green's equation. In fact, you can show using calculations virtually identical to what we just did that

$$G_{+}(t, \vec{r}; t', \vec{r}\,') = -\frac{1}{4\pi} \frac{\delta\left(t - t' + \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r} - \vec{r}\,'|}$$

is also a Green's function for the d'Alembertian. So which one do we use in finding the potentials? To answer that question requires physics, not maths, but we'll leave that until the next lecture because it's important enough to explain in a bit more detail that we want to do here.