MP465 – Advanced Electromagnetism

Problem Set 3

Due by 5pm on Thursday, 9 April 2020

1. The 3-dimensional Levi-Čivita symbol ϵ_{ijk}, where all indices range from 1 to 3 (or x to z), is defiined by

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any two indices are the same,} \\ +1 & \text{if } (ijk) \text{ is an even permutation of } (123), \\ -1 & \text{if } (ijk) \text{ is an odd permutation of } (123). \end{cases}$$

It satisfies the identity

$$\sum_{m=1}^{3} \epsilon_{ijm}\epsilon_{k\ell m} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}$$

and is used to define the cross-product via

$$\left(\vec{a} \times \vec{b}\right)_i = \sum_{j,k=1}^{3} \epsilon_{ijk} a_j b_k.$$

Using the above, prove the following two vector calculus identites:

(a) $\vec{a} \times \left(\vec{\nabla} \times \vec{b}\right) + \vec{b} \times \left(\vec{\nabla} \times \vec{a}\right) = \vec{\nabla}\left(\vec{a} \cdot \vec{b}\right) - \left(\vec{a} \cdot \vec{\nabla}\right)\vec{b} - \left(\vec{b} \cdot \vec{\nabla}\right)\vec{a},$

(b) $\vec{\nabla} \times \left(\vec{a} \times \vec{b}\right) = \vec{a}\left(\vec{\nabla} \cdot \vec{b}\right) + \left(\vec{b} \cdot \vec{\nabla}\right)\vec{a} - \vec{b}\left(\vec{\nabla} \cdot \vec{a}\right) - \left(\vec{a} \cdot \vec{\nabla}\right)\vec{b}.$

2. Suppose we have a linear medium subjected to an electromagnetic field, and we measure the magnitude of the fields inside the medium to be $15\,\mathrm{kV} \cdot \mathrm{m}^{-1}$ (which is about the strength of the electric field which would give you a slight zap when touching a metal doorknob) and $5\,\mathrm{mT}$ (the strength of the magnetic field of a typical fridge magnet). This fields will induce tiny electric and magnetic dipole moments in each constituent particle of the medium; determine the magnitudes of these moments for the following media: (a) water, (b) wood and (c)

air. (Assume that all media are at standard temperature and pressure, i.e. 20°C and one atmosphere).

Note: in order to do this problem, you must take it upon yourself to look up the necessary numbers like electric and magnetic susceptibilities, mass densities and the like, and you must cite your sources (as all good scientists must). Not doing so will result in marks being lost.

3. Maxwell's equations in matter are

$$\vec{\nabla}\cdot\vec{D}=\rho, \qquad \vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t},$$
$$\vec{\nabla}\cdot\vec{B}=0, \qquad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t},$$

where ρ and $\vec{J}$ are whatever free charge and current densities might be present. From these, show that for linear media, the energy-conservation equation

$$\frac{\partial u}{\partial t}+\vec{J}\cdot\vec{E} \;=\; -\vec{\nabla}\cdot\vec{S} \tag{1}$$

is satisfied if the electromagnetic (EM) field's energy density u and energy current (usually called the Poynting vector) $\vec{S}$ are given by, respectively,

$$u=\frac{1}{2}\left(\vec{D}\cdot\vec{E}+\vec{H}\cdot\vec{B}\right), \qquad \vec{S}=\vec{E}\times\vec{H}$$

Explanation: The first term in the left-hand side of (1) is the rate of change of the EM field's internal energy density; the second term is the rate of work per unit volume done on a free charge/current distribution by the EM field (note that the magnetic field does not contribute to this). Thus, the left-hand side is the rate at which the *total* energy density, of both the EM fields and the charge/current distribution, changes.

The right-hand side is an energy density flow rate due to propagation of the EM field. Thus, when integrated over a volume $\mathcal{V}$, this equation states that the energy flowing into $\mathcal{V}$ must equal the total rate of energy change due to both a change in the internal EM energy and work being done on the currents. Hence, energy is conserved for linear media.

4. The *Maxwell stress tensor* T is defined to be the 3×3 matrix with elements T_{ij} given by

$$T_{ij} \;:=\; \frac{1}{2}\left(\vec{D}\cdot\vec{E} + \vec{H}\cdot\vec{B}\right)\delta_{ij} - D_i E_j - H_i B_j.$$

Show that for a linear medium with permeability μ and permittivity ϵ, this quantity satisfies the momentum-conservation equation

$$\frac{\partial}{\partial t}\left(\mu\epsilon S_i\right) + f_i \;=\; -\sum_{j=1}^{3}\frac{\partial T_{ij}}{\partial x_j} \tag{2}$$

where

$$\vec{f} \;=\; \rho\vec{E} + \vec{J}\times\vec{B}$$

is the force per unit volume felt by the free charge/current distribution (obtained from the Lorentz force law).

Explanation: $\mu\epsilon\vec{S}$ is the momentum density of the EM field, so the first term on the left-hand side of (2) is its rate of change. The second term is the force per unit volume felt by the distribution; since force is the same as the rate of change of momentum, $\vec{f}$ is the rate at which the charge/current distribution's momentum density changes. The right-hand side is thus the rate of change of the *total* momentum density. Thus, following the same train of thought as in the previous problem, the right-hand side involving the stress tensor must be the rate at which the total momentum density is being carried away from the fields and charges (effectively, the force that's being exerted on its surroundings). Integrating over a volume $\mathcal{V}$ thus shows that the momentum flowing into $\mathcal{V}$ equals the momentum lost by $\mathcal{V}$. Ergo, momentum is conserved.

VECTOR CALCULUS FORMULAE

1. Cartesian coordinates (x, y, z) with constant unit direction vectors $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$

   - position vector: $\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$
   - line element: $\mathrm{d}\vec{r} = \mathrm{d}x\,\hat{e}_x + \mathrm{d}y\,\hat{e}_y + \mathrm{d}z\,\hat{e}_z$
     surface element: $\mathrm{d}\vec{\sigma} = \mathrm{d}y\,\mathrm{d}z\,\hat{e}_x + \mathrm{d}x\,\mathrm{d}z\,\hat{e}_y + \mathrm{d}x\,\mathrm{d}y\,\hat{e}_z$
     volume element: $\mathrm{d}^3\vec{r} = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$
   - gradient of a function $f(x, y, z)$:
   $$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{e}_x + \frac{\partial f}{\partial y}\hat{e}_y + \frac{\partial f}{\partial z}\hat{e}_z$$
   - divergence of a vector $\vec{A}(x, y, z) = A_x(x, y, z)\hat{e}_x + A_y(x, y, z)\hat{e}_y + A_z(x, y, z)\hat{e}_z$:
   $$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$
   - curl of a vector $\vec{A}(x, y, z) = A_x(x, y, z)\hat{e}_x + A_y(x, y, z)\hat{e}_y + A_z(x, y, z)\hat{e}_z$:
   $$\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{e}_z$$
   - Laplacian of a function $f(x, y, z)$:
   $$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

2. Cylindrical coordinates (r, ϕ, z) with unit direction vectors $\hat{e}_r$, $\hat{e}_\phi$, $\hat{e}_z$

   - relation to Cartesian coordinates: $x = r\cos\phi$, $y = r\sin\phi$, z unchanged
   - relation to Cartesian unit vectors:
   $$\left.\begin{array}{c} \hat{e}_r = \cos\phi\,\hat{e}_x + \sin\phi\,\hat{e}_y \\ \hat{e}_\phi = -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} \hat{e}_x = \cos\phi\,\hat{e}_r - \sin\phi\,\hat{e}_\phi \\ \hat{e}_y = \sin\phi\,\hat{e}_r + \cos\phi\,\hat{e}_\phi \end{array}\right.$$
   with $\hat{e}_z$ the same for both systems.

- position vector: $\vec{r} = r\hat{e}_r + z\hat{e}_z$
- line element: $\mathrm{d}\vec{r} = \mathrm{d}r\,\hat{e}_r + r\mathrm{d}\phi\,\hat{e}_\phi + \mathrm{d}z\,\hat{e}_z$
  surface element: $\mathrm{d}\vec{\sigma} = r\mathrm{d}\phi\,\mathrm{d}z\,\hat{e}_r + \mathrm{d}r\,\mathrm{d}z\,\hat{e}_\phi + r\mathrm{d}r\,\mathrm{d}\phi\,\hat{e}_z$
  volume element: $\mathrm{d}^3\vec{r} = r\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z$
- gradient of a function $f(r, \phi, z)$:

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{e}_\phi + \frac{\partial f}{\partial z}\hat{e}_z$$

- divergence of a vector $\vec{A}(r, \phi, z) = A_r(r, \phi, z)\hat{e}_r + A_\phi(r, \phi, z)\hat{e}_\phi + A_z(r, \phi, z)\hat{e}_z$:

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

- curl of a vector $\vec{A}(r, \phi, z) = A_r(r, \phi, z)\hat{e}_r + A_\phi(r, \phi, z)\hat{e}_\phi + A_z(r, \phi, z)\hat{e}_z$:

$$\vec{\nabla} \times \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{e}_\phi + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right)\hat{e}_z$$

- Laplacian of a function $f(r, \phi, z)$:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

3. Spherical coordinates (r, θ, ϕ) with unit direction vectors $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_\phi$

- relation to Cartesian coordinates: $x = r \sin\theta \cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$
- relation to Cartesian unit vectors:

$$\left.\begin{array}{c} \hat{e}_r = \sin\theta\cos\phi\,\hat{e}_x + \sin\theta\sin\phi\,\hat{e}_y + \cos\theta\,\hat{e}_z \\ \hat{e}_\theta = \cos\theta\cos\phi\,\hat{e}_x + \cos\theta\sin\phi\,\hat{e}_y - \sin\theta\,\hat{e}_z \\ \hat{e}_\phi = -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y \end{array}\right\}$$

$$\leftrightarrow \quad \left\{\begin{array}{c} \hat{e}_x = \sin\theta\cos\phi\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta - \sin\phi\,\hat{e}_\phi \\ \hat{e}_y = \sin\theta\sin\phi\,\hat{e}_r + \cos\theta\sin\phi\,\hat{e}_\theta + \cos\phi\,\hat{e}_\phi \\ \hat{e}_z = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta \end{array}\right.$$

- position vector: $\vec{r} = r\hat{e}_r$
- line element: $\mathrm{d}\vec{r} = \mathrm{d}r\,\hat{e}_r + r\mathrm{d}\theta\,\hat{e}_\theta + r\sin\theta\mathrm{d}\phi\,\hat{e}_\phi$
  surface element: $\mathrm{d}\vec{\sigma} = r^2\sin\theta\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{e}_r + r\sin\theta\mathrm{d}r\,\mathrm{d}\phi\,\hat{e}_\theta + r\mathrm{d}r\,\mathrm{d}\theta\,\hat{e}_\phi$
  volume element: $\mathrm{d}^3\vec{r} = r^2\sin\theta\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$
- gradient of a function $f(r, \theta, \phi)$:

$$\vec{\nabla} f \quad = \quad \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{e}_\phi$$

- divergence of a vector $\vec{A}(r,\theta,\phi) = A_r(r,\theta,\phi)\hat{e}_r + A_\theta(r,\theta,\phi)\hat{e}_\theta + A_\phi(r,\theta,\phi)\hat{e}_\phi$:

$$\vec{\nabla}\cdot\vec{A} \quad = \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

- curl of a vector $\vec{A}(r,\theta,\phi) = A_r(r,\theta,\phi)\hat{e}_r + A_\theta(r,\theta,\phi)\hat{e}_\theta + A_\phi(r,\theta,\phi)\hat{e}_\phi$:

$$\begin{aligned} \vec{\nabla}\times\vec{A} \quad = \quad & \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right)\hat{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right)\hat{e}_\theta \\ & + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{e}_\phi \end{aligned}$$

- Laplacian of a function $f(r,\theta,\phi)$:

$$\nabla^2 f \quad = \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$