MP465 – Advanced Electromagnetism

Problem Set 1

Due by 5pm on Thursday, 5 March 2020

1. Consider the following electrostatic potential, expressed in spherical coordinates:

$$\Phi(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r}+\frac{1}{a}\right)\exp\left(-\frac{2r}{a}\right).$$

where q is a constant and a is a positive constant. The charge density that gives this potential has the form

$$\rho(r,\theta,\phi) = A\delta^{(3)}(\vec{r}) + B(r)$$

for some constant A and some function $B(r)$. We wish to find both and interpret the results.

(a) For $r > 0$, only the second term in ρ appears. Using $\rho = -\epsilon_0\nabla^2\Phi$, compute it.

(b) The first term in ρ comes from what happens at $r = 0$. To find it, integrate ρ over a sphere of radius R centred at the origin and show that this integral does *not* vanish in the limit $R \to 0$ and thus determine the value of A. (Hint: remenber that $\nabla^2\Phi = \vec{\nabla}\cdot(\vec{\nabla}\Phi)$.)

(c) Now suppose $q = e$, the fundamental unit of electric charge (1.602×10^{-19} C, *not* Euler's constant $2.7182818\ldots$) and $a = a_0$, the Bohr radius (5.292×10^{-11} m). Given that the wavefunction for the ground state of a hydrogen atom is

$$\psi_{100}(\vec{r}) = \frac{1}{\sqrt{\pi a_0^3}}\exp\left(-\frac{r}{a_0}\right),$$

explain why this charge density is the correct one describing a quantum-mechanical hydrogen atom in its ground state (from which it immediately follows that Φ is the correct potential due to the entire atom).

2. A thin disc of radius a and constant surface charge density σ lies in the xy-plane with its centre at the origin. The density describing this disc in cylindrical coordinates is

$$\rho(r, \phi, z) = \sigma\delta(z)$$

for $r \leq a$, and zero otherwise. In tutorial, we found the potential $\Phi(r, \phi, z)$ and used it to show that the electric field along the z-axis was

$$\vec{E}(0, \phi, z) = \frac{\sigma}{2\epsilon_0}\left(\operatorname{sgn}(z) - \frac{z}{\sqrt{z^2 + a^2}}\right)\hat{e}_z.$$

However, we can also use the formula that gives $\vec{E}$ directly:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\vec{r}'\right)\left(\vec{r} - \vec{r}'\right)}{\left|\vec{r} - \vec{r}'\right|^3}\,\mathrm{d}^3\vec{r}'.$$

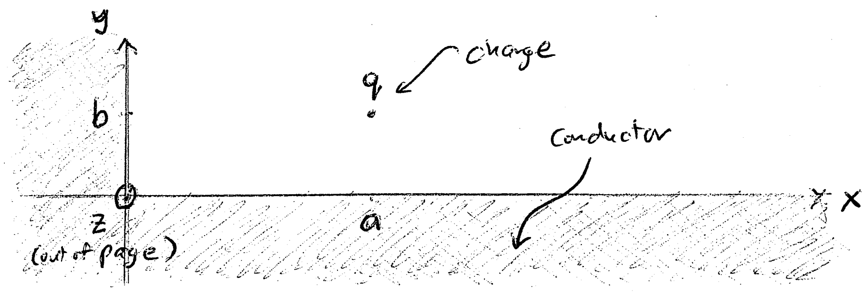
Show that using this gives the same result we obtained in tutorial.

3. A thin rod of length L and uniformly-distributed charge q lies along the z-axis with its centre at the origin.

   (a) Write down an expression for the density $\rho(r, \phi, z)$ as a function of the cylindrical coordinates (r, ϕ, z) in terms of the linear charge density $\lambda = q/L$.

   (b) Find the potential $\Phi(r, \phi, z)$.

4. A perfect conductor fills up a region of space given in Cartesian coordinates by $x < 0$, $y < 0$. A particle of charge q is placed near this conductor such that its position in Cartesian coordinates is $(a, b, 0)$, as shown below:



We find the potential for this system using the method of images as follows: two image charges, each with charge $-q$, are placed at

$(a, -b, 0)$ and $(-a, b, 0)$ and an image charge with charge q is placed at $(-a, -b, 0)$.

(a) Show that with these image charges included, the potential everywhere on the conductor's surface is zero.

(b) Compute the electric field in the region outside the conductor.

(c) Use your answer for (b) to find the surface charge density on the conductor and confirm that its integral over the surface is equal to the sum of all the image charges.

(Depending on how you do (c), the following identity may prove useful: for any real number z, $\arctan(z) + \arctan(1/z) = \pi/2$.)

VECTOR CALCULUS FORMULAE

1. Cartesian coordinates (x, y, z) with constant unit direction vectors $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$

   - position vector: $\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$
   - line element: $\mathrm{d}\vec{r} = \mathrm{d}x\,\hat{e}_x + \mathrm{d}y\,\hat{e}_y + \mathrm{d}z\,\hat{e}_z$
     surface element: $\mathrm{d}\vec{\sigma} = \mathrm{d}y\,\mathrm{d}z\,\hat{e}_x + \mathrm{d}x\,\mathrm{d}z\,\hat{e}_y + \mathrm{d}x\,\mathrm{d}y\,\hat{e}_z$
     volume element: $\mathrm{d}^3\vec{r} = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$
   - gradient of a function $f(x, y, z)$:

   $$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{e}_x + \frac{\partial f}{\partial y}\hat{e}_y + \frac{\partial f}{\partial z}\hat{e}_z$$

   - divergence of a vector $\vec{A}(x, y, z) = A_x(x, y, z)\hat{e}_x + A_y(x, y, z)\hat{e}_y + A_z(x, y, z)\hat{e}_z$:

   $$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

   - curl of a vector $\vec{A}(x, y, z) = A_x(x, y, z)\hat{e}_x + A_y(x, y, z)\hat{e}_y + A_z(x, y, z)\hat{e}_z$:

   $$\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{e}_z$$

   - Laplacian of a function $f(x, y, z)$:

   $$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

2. Cylindrical coordinates (r, ϕ, z) with unit direction vectors $\hat{e}_r$, $\hat{e}_\phi$, $\hat{e}_z$

   - relation to Cartesian coordinates: $x = r\cos\phi$, $y = r\sin\phi$, z unchanged
   - relation to Cartesian unit vectors:

   $$\left.\begin{array}{r} \hat{e}_r = \cos\phi\,\hat{e}_x + \sin\phi\,\hat{e}_y \\ \hat{e}_\phi = -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} \hat{e}_x = \cos\phi\,\hat{e}_r - \sin\phi\,\hat{e}_\phi \\ \hat{e}_y = \sin\phi\,\hat{e}_r + \cos\phi\,\hat{e}_\phi \end{array}\right.$$

   with $\hat{e}_z$ the same for both systems.

- position vector: $\vec{r} = r\hat{e}_r + z\hat{e}_z$
- line element: $\mathrm{d}\vec{r} = \mathrm{d}r\,\hat{e}_r + r\mathrm{d}\phi\,\hat{e}_\phi + \mathrm{d}z\,\hat{e}_z$
  surface element: $\mathrm{d}\vec{\sigma} = r\mathrm{d}\phi\,\mathrm{d}z\,\hat{e}_r + \mathrm{d}r\,\mathrm{d}z\,\hat{e}_\phi + r\mathrm{d}r\,\mathrm{d}\phi\,\hat{e}_z$
  volume element: $\mathrm{d}^3\vec{r} = r\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z$
- gradient of a function $f(r, \phi, z)$:

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{e}_\phi + \frac{\partial f}{\partial z}\hat{e}_z$$

- divergence of a vector $\vec{A}(r, \phi, z) = A_r(r, \phi, z)\hat{e}_r + A_\phi(r, \phi, z)\hat{e}_\phi + A_z(r, \phi, z)\hat{e}_z$:

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

- curl of a vector $\vec{A}(r, \phi, z) = A_r(r, \phi, z)\hat{e}_r + A_\phi(r, \phi, z)\hat{e}_\phi + A_z(r, \phi, z)\hat{e}_z$:

$$\vec{\nabla} \times \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{e}_\phi + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right)\hat{e}_z$$

- Laplacian of a function $f(r, \phi, z)$:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

3. Spherical coordinates (r, θ, ϕ) with unit direction vectors $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_\phi$

- relation to Cartesian coordinates: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$
- relation to Cartesian unit vectors:

$$\left.\begin{aligned} \hat{e}_r &= \sin\theta\cos\phi\,\hat{e}_x + \sin\theta\sin\phi\,\hat{e}_y + \cos\theta\,\hat{e}_z \\ \hat{e}_\theta &= \cos\theta\cos\phi\,\hat{e}_x + \cos\theta\sin\phi\,\hat{e}_y - \sin\theta\,\hat{e}_z \\ \hat{e}_\phi &= -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y \end{aligned}\right\}$$

$$\leftrightarrow \left\{\begin{aligned} \hat{e}_x &= \sin\theta\cos\phi\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta - \sin\phi\,\hat{e}_\phi \\ \hat{e}_y &= \sin\theta\sin\phi\,\hat{e}_r + \cos\theta\sin\phi\,\hat{e}_\theta + \cos\phi\,\hat{e}_\phi \\ \hat{e}_z &= \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta \end{aligned}\right.$$

- position vector: $\vec{r} = r\hat{e}_r$
- line element: $\mathrm{d}\vec{r} = \mathrm{d}r\,\hat{e}_r + r\mathrm{d}\theta\,\hat{e}_\theta + r\sin\theta\,\mathrm{d}\phi\,\hat{e}_\phi$
  surface element: $\mathrm{d}\vec{\sigma} = r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{e}_r + r\sin\theta\,\mathrm{d}r\,\mathrm{d}\phi\,\hat{e}_\theta + r\mathrm{d}r\,\mathrm{d}\theta\,\hat{e}_\phi$
  volume element: $\mathrm{d}^3\vec{r} = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$
- gradient of a function $f(r, \theta, \phi)$:

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{e}_\phi$$

- divergence of a vector $\vec{A}(r, \theta, \phi) = A_r(r, \theta, \phi)\hat{e}_r + A_\theta(r, \theta, \phi)\hat{e}_\theta + A_\phi(r, \theta, \phi)\hat{e}_\phi$:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

- curl of a vector $\vec{A}(r, \theta, \phi) = A_r(r, \theta, \phi)\hat{e}_r + A_\theta(r, \theta, \phi)\hat{e}_\theta + A_\phi(r, \theta, \phi)\hat{e}_\phi$:

$$\begin{aligned} \vec{\nabla}\times\vec{A} = {} & \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right)\hat{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right)\hat{e}_\theta \\ & + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{e}_\phi \end{aligned}$$

- Laplacian of a function $f(r, \theta, \phi)$:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$