Ollscoil Mhá Nuad
Maynooth University

SEMESTER 2

2019–2020

MP465
Advanced Electromagnetism

Prof. D. A. Johnston, Prof. P. Coles and Dr. P. Watts

Time allowed: 3 hours

Answer **ALL** questions

All questions carry equal marks

	Yes	No	N/A
Lecture and Tutorial notes allowed	✓		
Textbooks and Online References allowed	✓		
Consultation with other individuals allowed		✓	
Formula and Tables book allowed *(available at www.thphys.nuim.ie/Notes/MP465/)*	✓		
Nonprogrammable calculator allowed	✓		

1. A charged cube of side length a is centred at the origin of a Cartesian coordinate system such that the region of nonzero charge density is $-a/2 \leq x, y, z \leq a/2$. The $y < 0$ half of the cube has a charge density $-\rho_0$ and the $y > 0$ half has charge density $2\rho_0$, where ρ_0 is a positive constant.

   (a) Calculate the electric monopole and dipole moments of this system.

   [15 marks]

   (b) Determine the contributions of each of these moments to the multipole expansion of the scalar potential, expressing each as a function of (x, y, z).

   [10 marks]

2. A current circulates in a thin cylindrical shell of radius a and height h such that the current density is given in cylindrical coordinates by

$$\vec{J}(r, \phi, z) = \begin{cases} \frac{I}{h^2} z \delta(r - a) \hat{e}_\phi & \text{for } 0 \leq z \leq h, \\ 0 & \text{otherwise,} \end{cases}$$

   where I is a constant with units of current.

   (a) Show that the magnetic field on the z-axis is

$$\vec{B} = \frac{\mu_0 I}{2h^2} \left[\sqrt{z^2 + a^2} - \frac{z^2 + a^2 - hz}{\sqrt{(z-h)^2 + a^2}} \right] \hat{e}_z.$$

   [15 marks]

   (b) Calculate the cylinder's magnetic dipole moment.

   [10 marks]

   You may find the following integral useful for this problem:

$$\int \frac{\mathrm{d}s}{(s^2 + 1)^{3/2}} = \frac{s}{\sqrt{s^2 + 1}} + \text{constant.}$$

3. A circularly-polarised electromagnetic plane wave of frequency ω travelling through vacuum is normally incident on a linear dielectric material with index of refraction n and magnetic permeability $\mu = \mu_0$. If the boundary is at $z = 0$ and the wave travels in the positive z-direction, then the incident electric field is given (in complex form) by

$$\vec{E}_\mathrm{I}(t, \vec{r}) = \frac{1}{\sqrt{2}} \tilde{E}_0 e^{i(kz - \omega t)} \left(\hat{e}_x + i \hat{e}_y \right),$$

   where $\tilde{E}_0$ is the field's (complex) amplitude and $k = \omega/c$. Find the reflected and transmitted electric fields in terms of $\tilde{E}_0$, n and ω.

   [25 marks]

4. If S' is an inertial frame moving with velocity $\vec{v}$ relative to another inertial frame S, then the electric and magnetic fields measured in S' are related to the ones in S by

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}, \qquad \vec{E}'_{\perp} = \gamma(v)\left(\vec{E}_{\perp} + \vec{v} \times \vec{B}_{\perp}\right),$$
$$\vec{B}'_{\parallel} = \vec{B}_{\parallel}, \qquad \vec{B}'_{\perp} = \gamma(v)\left(\vec{B}_{\perp} - \frac{\vec{v}}{c^2} \times \vec{E}_{\perp}\right),$$

where $\parallel$ and $\perp$ denote, respectively, the components parallel and perpendicular to $\vec{v}$.

(a) Show that $\vec{E}' \cdot \vec{B}' = \vec{E} \cdot \vec{B}$ and $|\vec{E}'|^2 - c^2|\vec{B}'|^2 = |\vec{E}|^2 - c^2|\vec{B}|^2$. (You may assume the boost is in the x-direction.)

[15 marks]

(b) If $\vec{E} \cdot \vec{B} = 0$ but $|\vec{E}|^2 - c^2|\vec{B}|^2 < 0$, show there exists a frame in which there is no electric field. (**Hint:** think of a boost perpendicular to both $\vec{E}$ and $\vec{B}$.)

[10 marks]

MAXWELL'S EQUATIONS

- For electric field $\vec{E}$, displacement field $\vec{D}$, magnetic field $\vec{B}$, magnetic intensity $\vec{H}$, free charge density ρ and free current density $\vec{J}$:

$$\vec{\nabla} \cdot \vec{D} = \rho, \qquad \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t},$$
$$\vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

- Energy density and Poynting vector:

$$u = \frac{1}{2}\left(\vec{D} \cdot \vec{E} + \vec{H} \cdot \vec{B}\right), \qquad \vec{S} = \vec{E} \times \vec{H}.$$

VECTOR CALCULUS FORMULAE

1. Cartesian coordinates (x, y, z) with constant unit direction vectors $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$

   - position vector: $\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$
   - line element: $\mathrm{d}\vec{r} = \mathrm{d}x\,\hat{e}_x + \mathrm{d}y\,\hat{e}_y + \mathrm{d}z\,\hat{e}_z$
     surface element: $\mathrm{d}\vec{\sigma} = \mathrm{d}y\,\mathrm{d}z\,\hat{e}_x + \mathrm{d}x\,\mathrm{d}z\,\hat{e}_y + \mathrm{d}x\,\mathrm{d}y\,\hat{e}_z$
     volume element: $\mathrm{d}^3\vec{r} = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$
   - gradient of a scalar field $f(x, y, z)$:

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{e}_x + \frac{\partial f}{\partial y}\hat{e}_y + \frac{\partial f}{\partial z}\hat{e}_z$$

   - divergence of a vector field $\vec{A}(x, y, z) = A_x(x, y, z)\hat{e}_x + A_y(x, y, z)\hat{e}_y + A_z(x, y, z)\hat{e}_z$:

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

   - curl of a vector field $\vec{A}(x, y, z) = A_x(x, y, z)\hat{e}_x + A_y(x, y, z)\hat{e}_y + A_z(x, y, z)\hat{e}_z$:

$$\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{e}_z$$

   - Laplacian of a scalar field $f(x, y, z)$:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

2. Cylindrical coordinates (r, ϕ, z) with unit direction vectors $\hat{e}_r$, $\hat{e}_\phi$, $\hat{e}_z$

- relation to Cartesian coordinates: $x = r\cos\phi$, $y = r\sin\phi$, z unchanged
- relation to Cartesian unit vectors:

$$\left.\begin{aligned}\hat{e}_r &= \cos\phi\,\hat{e}_x + \sin\phi\,\hat{e}_y \\ \hat{e}_\phi &= -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y\end{aligned}\right\} \leftrightarrow \left\{\begin{aligned}\hat{e}_x &= \cos\phi\,\hat{e}_r - \sin\phi\,\hat{e}_\phi \\ \hat{e}_y &= \sin\phi\,\hat{e}_r + \cos\phi\,\hat{e}_\phi\end{aligned}\right.$$

  with $\hat{e}_z$ the same for both systems.
- position vector: $\vec{r} = r\hat{e}_r + z\hat{e}_z$
- line element: $\mathrm{d}\vec{r} = \mathrm{d}r\,\hat{e}_r + r\mathrm{d}\phi\,\hat{e}_\phi + \mathrm{d}z\,\hat{e}_z$
  surface element: $\mathrm{d}\vec{\sigma} = r\mathrm{d}\phi\,\mathrm{d}z\,\hat{e}_r + \mathrm{d}r\,\mathrm{d}z\,\hat{e}_\phi + r\mathrm{d}r\,\mathrm{d}\phi\,\hat{e}_z$
  volume element: $\mathrm{d}^3\vec{r} = r\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z$
- gradient of a scalar field $f(r, \phi, z)$:

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{e}_\phi + \frac{\partial f}{\partial z}\hat{e}_z$$

- divergence of a vector field $\vec{A}(r, \phi, z) = A_r(r, \phi, z)\hat{e}_r + A_\phi(r, \phi, z)\hat{e}_\phi + A_z(r, \phi, z)\hat{e}_z$:

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

- curl of a vector field $\vec{A}(r, \phi, z) = A_r(r, \phi, z)\hat{e}_r + A_\phi(r, \phi, z)\hat{e}_\phi + A_z(r, \phi, z)\hat{e}_z$:

$$\vec{\nabla} \times \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{e}_\phi + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right)\hat{e}_z$$

- Laplacian of a scalar field $f(r, \phi, z)$:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

3. Spherical coordinates (r, θ, ϕ) with unit direction vectors $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_\phi$

- relation to Cartesian coordinates: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$
- relation to Cartesian unit vectors:

$$\left.\begin{array}{c} \hat{e}_r = \sin\theta\cos\phi\,\hat{e}_x + \sin\theta\sin\phi\,\hat{e}_y + \cos\theta\,\hat{e}_z \\ \hat{e}_\theta = \cos\theta\cos\phi\,\hat{e}_x + \cos\theta\sin\phi\,\hat{e}_y - \sin\theta\,\hat{e}_z \\ \hat{e}_\phi = -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y \end{array}\right\}$$

$$\leftrightarrow \left\{\begin{array}{c} \hat{e}_x = \sin\theta\cos\phi\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta - \sin\phi\,\hat{e}_\phi \\ \hat{e}_y = \sin\theta\sin\phi\,\hat{e}_r + \cos\theta\sin\phi\,\hat{e}_\theta + \cos\phi\,\hat{e}_\phi \\ \hat{e}_z = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta \end{array}\right.$$

- position vector: $\vec{r} = r\hat{e}_r$
- line element: $\mathrm{d}\vec{r} = \mathrm{d}r\,\hat{e}_r + r\mathrm{d}\theta\,\hat{e}_\theta + r\sin\theta\mathrm{d}\phi\,\hat{e}_\phi$
  surface element: $\mathrm{d}\vec{\sigma} = r^2\sin\theta\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{e}_r + r\sin\theta\mathrm{d}r\,\mathrm{d}\phi\,\hat{e}_\theta + r\mathrm{d}r\,\mathrm{d}\theta\,\hat{e}_\phi$
  volume element: $\mathrm{d}^3\vec{r} = r^2\sin\theta\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$
- gradient of a scalar field $f(r, \theta, \phi)$:

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{e}_\phi$$

- divergence of a vector field $\vec{A}(r,\theta,\phi) = A_r(r,\theta,\phi)\hat{e}_r + A_\theta(r,\theta,\phi)\hat{e}_\theta + A_\phi(r,\theta,\phi)\hat{e}_\phi$:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

- curl of a vector field $\vec{A}(r,\theta,\phi) = A_r(r,\theta,\phi)\hat{e}_r + A_\theta(r,\theta,\phi)\hat{e}_\theta + A_\phi(r,\theta,\phi)\hat{e}_\phi$:

$$\begin{aligned}\vec{\nabla}\times\vec{A} &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right)\hat{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right)\hat{e}_\theta \\ &\quad + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{e}_\phi\end{aligned}$$

- Laplacian of a scalar field $f(r, \theta, \phi)$:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$