Solution to MP465 Exam, 2019-20

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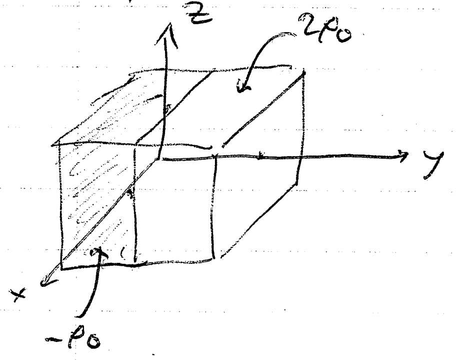
P.1



The configuration described is pictured to the left, and we want to find its electric monopole and dipole moments, as well as their contributions to the overall electric scalar potential.

1a) The electric monopole is simply the total charge. The $y<0$ block has a constant density $\rho = -\rho_0$ and a total volume of $\frac{1}{2}a^3$, so $q_{y<0} = (-\rho_0)(\frac{1}{2}a^3) = -\frac{1}{2}\rho_0 a^3$. The $y>0$ half has $\rho = 2\rho_0$ and is also $\frac{1}{2}a^3$ in volume, so $q_{y>0} = (2\rho_0)(\frac{1}{2}a^3) = \rho_0 a^3$.

Thus,

$$q = q_{y<0} + q_{y>0} = \boxed{\frac{1}{2}\rho_0 a^3}$$ [5 marks]

is the monopole moment.

The electric dipole moment is given by the formula $\vec{p} = \int \rho(\vec{r}')\vec{r}'d^3\vec{r}'$,

or more properly,

$p_x = \int \rho(x',y',z')x'dx'dy'dz'$, $p_y = \int \rho(x',y',z')y'dx'dy'dz'$, $p_z = \int \rho(x',y',z')z'dx'dy'dz'$!

Thus,

$$p_x = \int \rho(x',y',z')\,x'dx'dy'dz' = \left(\int_{-a/2}^{a/2} x'dx'\right)\left(\int_{-a/2}^{a/2} \rho\, dy'\right)\left(\int_{-a/2}^{a/2} dz'\right)$$

$$= 0$$

since x' is an odd function. Similarly, $p_z = 0$ since $\int_{-a/2}^{a/2} z'dz' = 0$. Thus, the only nonzero component is p_y:

$$p_y = \left(\int_{-a/2}^{a/2} dx'\right)\left(\int_{-a/2}^{a/2} \rho y'dy'\right)\left(\int_{-a/2}^{a/2} dz'\right)$$

$$= a^2\left[\int_{-a/2}^{0} \rho y'dy' + \int_{0}^{a/2} \rho y'dy'\right]$$

$$= a^2\left[-\rho_0\int_{-a/2}^{0} y'dy' + 2\rho_0\int_{0}^{a/2} y'dy'\right]$$

$$= a^2\left[-\rho_0\left(-\frac{a^2}{4}\right) + 2\rho_0\left(\frac{a^2}{4}\right)\right] = 3\rho_0 a^4/4$$

and thus

$$\boxed{\vec{p} = \frac{3\rho_0 a^4}{4}\hat{e}_y}$$ [10 marks]

(b) The monopole and dipole contributions to the scalar potential Φ are

$$\Phi_0 = \frac{1}{4\pi\epsilon_0}\frac{q}{r}, \quad \Phi_1 = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$$

so we have

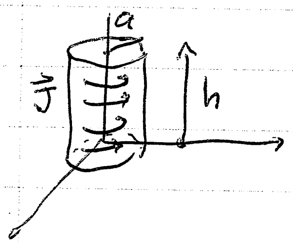
$$\Phi_0(x,y,z) = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{x^2+y^2+z^2}} = \boxed{\frac{\rho_0 a^3}{8\pi\epsilon_0}\frac{1}{\sqrt{x^2+y^2+z^2}}}$$ [5 marks]

and

$$\Phi_1(x,y,z) = \frac{1}{4\pi\epsilon_0}\frac{\left(\frac{3\rho_0 a^4}{4}\hat{e}_y\right)\cdot(x\hat{e}_x+y\hat{e}_y+z\hat{e}_z)}{(x^2+y^2+z^2)^{3/2}}$$

$$= \boxed{\frac{3\rho_0 a^4}{16\pi\epsilon_0}\frac{y}{(x^2+y^2+z^2)^{3/2}}}$$ [5 marks]

P. 2



The shell is depicted to the left, with $\vec{J} = \frac{Iz}{h^2}\delta(r-a)\hat{e}_\phi$.

(a) The general form (i.e. the generalized Biot-Savart formula) is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}d^3\vec{r}\,'$$

For arbitrary $\vec{r}$, we would need elliptic integrals to describe this, but we only want $\vec{B}$ on the z-axis, which is $r=0$ in cylindrical coordinates. Thus, $\vec{r} = z\hat{e}_z$ and so

$$\vec{r}-\vec{r}\,' = -r'\hat{e}_{r'} + (z-z')\hat{e}_z$$

$$|\vec{r}-\vec{r}\,'| = \sqrt{(r')^2+(z'-z)^2}$$

giving

$$\vec{B}(0,\phi,z) = \frac{\mu_0}{4\pi}\int \frac{\frac{I}{h^2}z'\delta(r'-a)\hat{e}_{\phi'}\times(-r'\hat{e}_{r'}+(z-z')\hat{e}_z)}{[(r')^2+(z'-z)^2]^{3/2}}\,r'dr'd\phi'dz'$$

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$$= \frac{\mu_0 I}{4\pi h^2}\int \frac{z'\delta(r'-a)[(z-z')\hat{e}_{r'}+r'\hat{e}_z]}{[(r')^2+(z'-z)^2]^{3/2}}\, r'dr'd\phi'dz'$$

Doing the r' integral replaces r' by a everywhere (due to the δ-function):

$$\vec{B}(0,\phi,z) = \frac{\mu_0 I a}{4\pi h^2}\int \frac{z'(z-z')\hat{e}_{r'} + az'\hat{e}_z}{[(z'-z)^2+a^2]^{3/2}}\, d\phi' dz'.$$

The only ϕ' dependence in the integrand is in $\hat{e}_{r'} = \cos\phi'\hat{e}_x + \sin\phi'\hat{e}_y$, so because $\int_0^{2\pi}\cos\phi' d\phi' = \int_0^{2\pi}\sin\phi' d\phi' = 0$, only the z-component remains (using $\int_0^{2\pi} d\phi' = 2\pi$):

$$\vec{B}(0,\phi,z) = \frac{\mu_0 I a^2 \hat{e}_z}{2h^2}\int_0^h \frac{z'}{[(z'-z)^2+a^2]^{3/2}}\, dz'$$

Let $s = \frac{z'-z}{a}$ be a new integration variable. We then have

$$\vec{B}(0,\phi,z) = \frac{\mu_0 I a^2}{2h^2}\int_{-z/a}^{\frac{h-z}{a}} \frac{(as+z)}{a^3(s^2+1)^{3/2}}\, a\,ds \quad \hat{e}_z$$

$$= \frac{\mu_0 I}{2h^2}\left[a\int_{-z/a}^{\frac{h-z}{a}} \frac{s}{(s^2+1)^{3/2}} + \int_{-z/a}^{\frac{h-z}{a}} \frac{1}{(s^2+1)^{3/2}}\, ds \right]\hat{e}_z$$

$$= \frac{\mu_0 I}{2h^2}\left[-\frac{a}{\sqrt{s^2+1}} + z\frac{s}{\sqrt{s^2+1}} \right]_{-z/a}^{\frac{h-z}{a}} \hat{e}_z$$

where we used the integral provided. Thus,

$$\vec{B}(0,\phi,z) = \frac{\mu_0 I}{2h^2}\left[-\frac{a}{\sqrt{(\frac{h-z}{a})^2+1}} + z\frac{\frac{h-z}{a}}{\sqrt{(\frac{h-z}{a})^2+1}} + \frac{a}{\sqrt{\frac{z^2}{a^2}+1}} - z\frac{(-z/a)}{\sqrt{\frac{z^2}{a^2}+1}} \right]\hat{e}_z$$

$$= \frac{\mu_0 I}{2h^2}\left[\frac{a(\frac{z^2}{a^2}+1)}{\sqrt{\frac{z^2}{a^2}+1}} - \frac{z(\frac{z-h}{a})+a}{\sqrt{(\frac{z-h}{a})^2+1}} \right]\hat{e}_z$$

$$= \boxed{\frac{\mu_0 I}{2h^2}\left[\sqrt{z^2+a^2} - \frac{z^2+a^2-hz}{\sqrt{(z-h)^2+a^2}} \right]\hat{e}_z}$$

[15 marks]

as desired.

(b) The magnetic dipole moment is obtained from the formula

$$\vec{m} = \frac{1}{2}\int \vec{r}'\times\vec{J}(\vec{r}')\, d^3\vec{r}'$$

so for this case,

$$\vec{m} = \frac{1}{2}\int (r'\hat{e}_{r'} + z'\hat{e}_{z'})\times\left(\frac{I}{h^2}z'\delta(r'-a)\hat{e}_{\phi'}\right) r'dr'd\phi'dz'$$

$$= \frac{I}{2h^2}\int z'\,\delta(r'-a)(-z'\vec{e}_{r'} + r'\vec{e}_z)\,r'dr'd\phi'dz'$$

$$= \frac{Ia}{2h^2}\int\int\left[-(z')^2\hat{e}_{r'} + z'a\,\hat{e}_z\right]d\phi'dz'$$

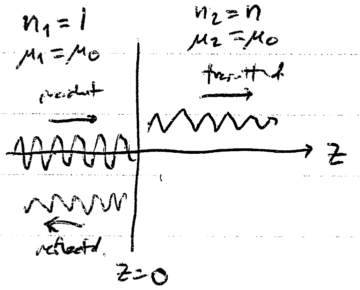
after doing the r' integral. Again, $\int_0^{2\pi}\hat{e}_{r'}\,d\phi' = \vec{0}$ and $\int_0^{2\pi}d\phi' = 2\pi$ leaves only a z-component to give

$$\vec{m} = \frac{\pi I a^2 \vec{e}_z}{h^2}\int_0^h z'dz' = \frac{\pi I a^2}{h^2}\vec{e}_z\left(\tfrac{1}{2}h^2\right) = \boxed{\tfrac{1}{2}\pi I a^2\vec{e}_z} \quad [10 \text{ marks}]$$

as the cylinder's magnetic dipole moment.

P.3

Recall the basic rules for an EM wave at a boundary between two dielectric materials: $D_\perp$, $\vec{E}_\parallel$, $B_\perp$ and $\vec{H}_\parallel$ are all continuous, where $\perp$ indicates the component [5 marks] normal to the boundary and $\parallel$ the two components parallel to the boundary.



Note here that the wave is normally incident, i.e. in the z-direction. Since $\vec{E}$ and $\vec{B}$ are both $\perp$ to z, so are $\vec{D} = \epsilon\vec{E}$ and $\vec{H} = \frac{1}{\mu}\vec{B}$ and thus $D_\perp$ and $B_\perp$ are automatically zero, so their continuity is automatic. Since $\vec{E}_\parallel = \vec{E}$, this means $\vec{E}$ is continuous across $z=0$, and since $\mu_1 = \mu_2 = 0$, $\vec{H}_\parallel$ continuous $\Rightarrow \frac{1}{\mu_0}\vec{B}_\parallel$ continuous $\Rightarrow \vec{B}$ continuous. So for this case, E_x, E_y, B_x and B_y are all continuous across the boundary.

We're allowed to assume $\theta_R = \theta_I$ and $n_1\sin\theta_I = n_2\sin\theta_T$ in all cases, so since $\theta_I = 0$ ("normal incidence"), $\theta_R = \theta_T = 0$ follow, so the fields for $z<0$ have the following forms:

$$\vec{E}_I = \frac{1}{\sqrt{2}}\tilde{E}_0 e^{i(kz-\omega t)}(\hat{e}_x + i\vec{e}_y) \qquad k = \omega/c$$

$$\vec{E}_R = (\tilde{E}_{Rx}\vec{e}_x + \tilde{E}_{Ry}\hat{e}_y)e^{i(-kz-\omega t)}$$

since $k_R = -k\hat{e}_z$. For $z>0$, $k_T = \frac{\omega}{(c/n)}\vec{e}_z = \frac{n\omega}{c}\vec{e}_z$ since the speed of light is c/n in this medium, thus

$$E_T = (\tilde{E}_{Tx}\vec{e}_x + \tilde{E}_{Ty}\vec{e}_y)e^{i(nkz-\omega t)}.$$

We know $\vec{B} = \frac{\vec{k}}{\omega}\times\vec{E}$, so the magnetic fields are

$$\vec{B}_I = \frac{(\omega/c)\vec{e}_z}{\omega}\times\frac{1}{\sqrt{2}}\tilde{E}_0 e^{i(kz-\omega t)}(\vec{e}_x + i\vec{e}_y) = \frac{1}{c\sqrt{2}}\tilde{E}_0(-i\vec{e}_x + \vec{e}_y)e^{i(kz-\omega t)}$$

$$\vec{B}_R = \frac{\left(-\frac{\omega}{c}\vec{e}_z\right)}{\omega} \times \left(\tilde{E}_{Rx}\vec{e}_x + \tilde{E}_{Ry}\vec{e}_y\right) e^{i(-kz-\omega t)} = \frac{1}{c}\left(\tilde{E}_{Ry}\vec{e}_x - \tilde{E}_{Rx}\vec{e}_y\right) e^{i(-kz-\omega t)}$$

$$\vec{B}_T = \frac{\left(\frac{n\omega}{c}\vec{e}_z\right)}{\omega} \times \left(\tilde{E}_{Tx}\vec{e}_x + \tilde{E}_{Ty}\vec{e}_y\right) e^{i(nkz-\omega t)} = \frac{n}{c}\left(-\tilde{E}_{Ty}\vec{e}_x + \tilde{E}_{Tx}\vec{e}_y\right) e^{i(nkz-\omega t)}$$

so continuity at $z=0$ (and for all time t) means

$$\vec{E}_I + \vec{E}_R = \vec{E}_T \Rightarrow \frac{1}{\sqrt{2}}E_0 + \tilde{E}_{Rx} = \tilde{E}_{Tx}, \quad \frac{i}{\sqrt{2}}E_0 + \tilde{E}_{Ry} = \tilde{E}_{Ty}$$ [5 marks]

$$\vec{B}_I + \vec{B}_R = \vec{B}_T \Rightarrow -\frac{i}{c\sqrt{2}}E_0 + \frac{1}{c}\tilde{E}_{Ry} = -\frac{n}{c}\tilde{E}_{Ty}, \quad \frac{1}{c\sqrt{2}}E_0 - \frac{1}{c}\tilde{E}_{Rx} = \frac{n}{c}\tilde{E}_{Tx}$$ [5 marks]

which gives us four eqns for four unknowns:

$$\left.\begin{array}{l} \tilde{E}_{Rx} - \tilde{E}_{Tx} = -\frac{1}{\sqrt{2}}E_0 \\ \tilde{E}_{Rx} + n\tilde{E}_{Tx} = \frac{1}{\sqrt{2}}E_0 \end{array}\right\} \Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & n \end{pmatrix}\begin{pmatrix} \tilde{E}_{Rx} \\ \tilde{E}_{Tx} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}E_0$$

$$\left.\begin{array}{l} \tilde{E}_{Ry} - \tilde{E}_{Ty} = -\frac{i}{\sqrt{2}}E_0 \\ \tilde{E}_{Ry} + n\tilde{E}_{Ty} = \frac{i}{\sqrt{2}}E_0 \end{array}\right\} \Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & n \end{pmatrix}\begin{pmatrix} \tilde{E}_{Ry} \\ \tilde{E}_{Ty} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}\frac{i}{\sqrt{2}}E_0$$

so

$$\begin{pmatrix} \tilde{E}_{Rx} \\ \tilde{E}_{Tx} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & n \end{pmatrix}^{-1}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}E_0 = \frac{1}{n+1}\begin{pmatrix} n & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}E_0 = \begin{pmatrix} \frac{1-n}{1+n} \\ \frac{2}{1+n} \end{pmatrix}\frac{1}{\sqrt{2}}E_0$$

and

$$\begin{pmatrix} \tilde{E}_{Ry} \\ \tilde{E}_{Ty} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & n \end{pmatrix}^{-1}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\frac{i}{\sqrt{2}}E_0 = \begin{pmatrix} \frac{1-n}{1+n} \\ \frac{2}{1+n} \end{pmatrix}\frac{i}{\sqrt{2}}E_0$$

so $E_{Rx} = \frac{1}{\sqrt{2}}\left(\frac{1-n}{1+n}\right)E_0$, $E_{Ry} = \frac{i}{\sqrt{2}}\left(\frac{1-n}{1+n}\right)E_0$, $E_{Tx} = \frac{\sqrt{2}}{1+n}E_0$, $E_{Ty} = \frac{i\sqrt{2}}{1+n}E_0$

giving

$$\boxed{\vec{E}_R = \frac{1}{\sqrt{2}}\left(\frac{1-n}{1+n}\right)E_0\left(\vec{e}_x + i\vec{e}_y\right)e^{-\frac{i\omega}{c}(z+ct)}}$$ [5 marks]

$$\boxed{\vec{E}_T = \left(\frac{\sqrt{2}}{1+n}\right)E_0\left(\vec{e}_x + i\vec{e}_y\right)e^{\frac{in\omega}{c}\left(z+\frac{c}{n}t\right)}}$$ [5 marks]

P.4

(a) For a boost in the x-direction, $\vec{v} = v\hat{e}_x$, so since $\parallel = x$, we see $E'_x = E_x$ and $B'_x = B_x$. Since $\vec{E}_\perp = E_y\hat{e}_y + E_z\hat{e}_z$ and $\vec{B}_\perp = B_y\hat{e}_y + B_z\hat{e}_z$, we find

$$\vec{E}_\perp{}' = \gamma(v)\left(\vec{E}_\perp + \vec{v}\times\vec{B}_\perp\right) \Rightarrow E_y' = \gamma(v)(E_y - vB_z),\ E_z' = \gamma(v)(E_z + vB_y)$$

$$\vec{B}_\perp{}' = \gamma(v)\left(\vec{B}_\perp - \frac{\vec{v}}{c^2}\times\vec{E}_\perp\right) \Rightarrow B_y' = \gamma(v)\left(B_y + \frac{v}{c^2}E_z\right),\ B_z' = \gamma(v)\left(B_z - \frac{v}{c^2}E_y\right).$$

Thus,

$$\vec{E}'\cdot\vec{B}' = E_x'B_x' + E_y'B_y' + E_z'B_z'$$

$$= E_xB_x + \gamma^2(v)\left[(E_y - vB_z)\left(B_y + \frac{v}{c^2}E_z\right) + (E_z + vB_y)\left(B_z - \frac{v}{c^2}E_y\right)\right]$$

$$= E_xB_x + \gamma^2(v)\left[E_yB_y - vB_yB_z + \frac{v}{c^2}E_yE_z - \frac{v^2}{c^2}B_zE_z + E_zB_z + vB_yB_z - \frac{v}{c^2}E_yE_z - \frac{v^2}{c^2}B_yE_y\right]$$

$$= E_xB_x + \gamma^2(v)\left(1 - \frac{v^2}{c^2}\right)(E_yB_y + E_zB_z) = E_xB_x + E_yB_y + E_zB_z$$

$$= \boxed{\vec{E}\cdot\vec{B}}$$

[7 marks]

because $\gamma^2(v) = \frac{1}{1 - v^2/c^2}$. Now for the other one:

$$|\vec{E}'|^2 - c^2|\vec{B}'|^2 = (E_x')^2 + (E_y')^2 + (E_z')^2 - c^2(B_x')^2 - c^2(B_y')^2 - c^2(B_z')^2$$

$$= E_x^2 + \gamma^2(v)\left[(E_y - vB_z)^2 + (E_z + vB_y)^2\right] - c^2B_x^2 - c^2\gamma^2(v)\left[\left(B_y + \frac{v}{c^2}E_z\right)^2 + \left(B_z - \frac{v}{c^2}E_y\right)^2\right]$$

$$= E_x^2 - c^2B_x^2 + \gamma^2(v)\left[E_y^2 - 2vE_yB_z + v^2B_z^2 + E_z^2 + 2vE_zB_y + v^2B_y^2\right] - c^2\gamma^2(v)\left[B_y^2 + \frac{2v}{c^2}E_zB_y + \frac{v^2}{c^4}E_z^2 + B_z^2 - \frac{2v}{c^2}E_yB_z + \frac{v^2}{c^4}E_y^2\right]$$

$$= E_x^2 - c^2B_x^2 + \gamma^2(v)\left[\left(1 - \frac{v^2}{c^2}\right)(E_y^2 + E_z^2) + (v^2 - c^2)(B_y^2 + B_z^2)\right]$$

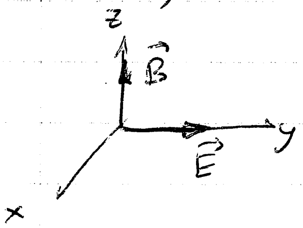
$$= E_x^2 + E_y^2 + E_z^2 - c^2(B_x^2 + B_y^2 + B_z^2)$$

$$= \boxed{|\vec{E}|^2 - c^2|\vec{B}|^2}$$

[8 marks]

as desired. Thus, $\vec{E}\cdot\vec{B}$ and $|\vec{E}|^2 - c^2|\vec{B}|^2$ are both Lorentz-invariant.

(b) $\vec{E}\cdot\vec{B}=0$ means $\vec{E}$ and $\vec{B}$ are perpendicular to each other. Without loss of generality, we can define a coordinate system such that $\vec{E}$ lies along the y axis and $\vec{B}$ along the z-axis, as shown below. The hint tells us we should then consider a boost in the direction perpendicular to both, i.e. in the x-direction.



We already have the transformation rules from (a), so since $E_x = E_z = 0$ and $B_y = B_z = 0$, $\vec{E} = E\hat{e}_y$ and $\vec{B} = B\hat{e}_z$, with E and B both positive. Then,

$E_x' = 0$, $E_y' = \gamma(v)(E - vB)$, $E_z' = 0$

$B_x' = 0$, $B_y' = 0$, $B_z' = \gamma(v)(B - \frac{v}{c^2}E)$ [Similar]

are the fields in the boosted frame. Now, the condition $|\vec{E}|^2 - c^2|\vec{B}|^2 < 0$ means $|\vec{E}| < c|\vec{B}|$, or $E < cB$. Now, if v is such that the boosted frame S' has $\vec{E}' = \vec{0}$, then $\gamma(v)(E - vB) = 0$, or $E = vB$. Since $E < cB$, this implies $vB < cB \Rightarrow v < c$, i.e. a physically permissible boost. Thus, if $\vec{v} = |\vec{E}|/|\vec{B}|\,\hat{e}_x$, then $\boxed{\text{we have a frame in which } \vec{E}' = \vec{0}.}$ [5 marks]

(Note that $\vec{B}' = \vec{0}$ is not possible: $\vec{B}' = \vec{0} \Rightarrow B = \frac{v}{c^2}E$, but $E < cB$ gives $B < \frac{v}{c}B \Rightarrow v > c$, impossible. So $|\vec{E}|^2 - c^2|\vec{B}|^2 < 0$ means there is a nonzero magnetic field in every inertial frame!)