MP465 – Advanced Electromagnetism

Lectures 23 & 24 (6 May 2020)

C. The Field Strength Tensor and Transformation Law for the Electromagnetic Field

Last time, we realised that the scalar and vector potentials can be put together into a 4-vector A^μ as $A^0 = \Phi/c$, $A^{1,2,3} = A_{x,y,z}$. However, the electric and magnetic fields are six objects and it's probably not obvious how these can be put into a relativistic context. However, if we write these fields in terms of the components of the 4-potential, maybe we can get an idea of how to proceed.

We look first at the x-component of the electric field: it's

$$E_x = -\frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{\partial t}$$

so if we write the potentials and derivatives in terms of the components of A^μ and ∂_μ, we get

$$\begin{aligned} E_x &= -\left(\partial_1\right)\left(cA^0\right) - \left(c\partial_0\right)\left(A^1\right) \\ &= c\left(\partial_1 A_0 - \partial_0 A_1\right). \end{aligned}$$

Recall that, because of the metric we're using, any time we change the position of a time (0) index we pick up a factor of -1, and when we change the position of a spacial (1,2 or 3) index, the sign doesn't change. That's why we have $A^0 = -A_0$ and $A^1 = A_1$ in the above. If we do the same for the y- and z-components of $\vec{E}$, virtually identical calculations give

$$E_y = c\left(\partial_2 A_0 - \partial_0 A_2\right), \qquad E_z = c\left(\partial_3 A_0 - \partial_0 A_3\right).$$

Now for the magnetic field: the x-component is

$$\begin{aligned} B_x &= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ &= \left(\partial_2\right)\left(A^3\right) - \left(\partial_3\right)\left(A^2\right) \\ &= \partial_2 A_3 - \partial_3 A_3 \end{aligned}$$

and similar calculations give the other two components as

$$B_y = \partial_3 A_1 - \partial_1 A_3, \qquad B_z = \partial_1 A_2 - \partial_2 A_1.$$

And now we see a pattern emerging: the electric and magnetic fields are given by antisymmetric combinations of derivatives acting on potentials. This suggests the definition of a new quantity, called the *field strength tensor* (sometimes just "field strength" for brevity), as

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu .$$

Note that this is antisymmetric under interchange of the two indices, so we have $F_{\nu\mu} = -F_{\mu\nu}$ and thus $F_{00} = F_{11} = F_{22} = F_{33} = 0$. Therefore, there are only six independent components, which is precisely what we need to accommodate the electric and magnetic fields:

$$\begin{aligned}
&F_{01} = -F_{10} = -E_x/c,\; F_{02} = -F_{20} = -E_y/c,\; F_{03} = -F_{30} = -E_z/c,\\
&F_{12} = -F_{21} = B_z,\; F_{23} = -F_{32} = -B_x,\; F_{31} = -F_{13} = B_y.
\end{aligned}$$

For purposes of brevity and calculation, it's standard to think these as the components of an antisymmetric 4×4 matrix:

$$\begin{aligned}
F_{\mu\nu} &= \begin{pmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}.
\end{aligned}$$

We can, of course, use the usual rules of index manipulation to raise one or both indices of this. For example $F^{11} = F_1{}^1 = F_{11} = 0$, $F^{03} = F^0{}_3 = -F_{03} = E_z/c$ and $F^{12} = F_1{}^2 = F_{12} = B_z$. The field strength with two upper indices, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, is therefore

$$F^{\mu\nu} \;=\; \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}.$$

We also see a very nice way to express the gauge invariance of these fields: recall that we showed that if Φ and $\vec{A}$ lead to the fields $\vec{E}$ and $\vec{B}$, then so do $\Phi - \partial\chi/\partial t$ and $\vec{A} + \vec{\nabla}\chi$. This is gauge invariance, and is what allows flexibility in our choice of potentials. Now, notice that

$$\begin{aligned}
\Phi - \frac{\partial \chi}{\partial t} &= cA^0 - c\partial_0\chi \\
&= c(A^0 + \partial^0\chi), \\
(\vec{A} + \vec{\nabla}\chi)_x &= A^1 + \partial_1\chi \\
&= A^1 + \partial^1\chi
\end{aligned}$$

and similarly for the y- and z-components of $\vec{A} + \vec{\nabla}\chi$. So we see that a gauge transformation may be written in 4-vector notation as $A^\mu \mapsto A^\mu + \partial^\mu\chi$.

When we compute the field strength tensor using this transformed potential, we get

$$\begin{aligned}
\partial^\mu\left(A^\nu + \partial^\nu\chi\right) - \partial^\nu\left(A^\mu + \partial^\mu\chi\right) &= \partial^\mu A^\nu - \partial^\nu A^\mu + \partial^\mu\partial^\nu\chi - \partial^\nu\partial^\mu\chi \\
&= F^{\mu\nu}
\end{aligned}$$

because partial derivatives can be done in any order and thus the final two terms cancel. Since F doesn't change under this transformation, its components – the electric and magnetic fields – are also the same and thus gauge invariance holds.

Now, we're in a position to see how the electric and magnetic fields change between inertial frames, but an important comment: these are *not* Lorentz transformations. A Lorentz transformation is only for 4-vectors, and the electric and magnetic fields are *not* 4-vectors. However, we can use the field strength tensor to determine how they do transform.

How? The field strength tensor is explicitly constructed from two 4-vectors – the derivative and potential 4-vectors – and we know how they transform, so we also know how the field strength tensor transforms. Specifically, because anything with an upper index transforms the same way as the coordinate 4-vector, namely, $x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$, this means the field strength tensor with two upper indices changes between inertial frames as

$$F^{\mu\nu} \quad \mapsto \quad F'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}.$$

This may look rather formidable, but there's a way to rewrite it to make the calculation (somewhat) easier: let F be the 4×4 matrix with components $F^{\mu\nu}$. Now, recall the definition of matrix multiplication: if M and N are two

$n \times n$ matrices, then the matrix MN has components

$$(MN)_{ij} = \sum_{k=1}^{n} M_{ik} N_{kj}.$$

So we see that any time we sum over an index which is the second of one matrix and the first of another (as k is above), that's a matrix multiplication. So in our expression for F' above, we see part of it is $\Lambda^\mu{}_\alpha F^{\alpha\beta}$ (with an implied sum over α). Thus, if Λ is the 4×4 matrix giving our Lorentz transformation, this quantity is $(\Lambda F)^{\mu\beta}$. There's the other Λ though, but we see that $\Lambda^\nu{}_\beta$ is not only the $\nu\beta$-element of the matrix Λ but also the $\beta\nu$-element of the transposed matrix Λ^T. Thus

$$\begin{aligned}(F')^{\mu\nu} &= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta} \\ &= (\Lambda F)^{\mu\beta} (\Lambda^T)_\beta{}^\nu \\ &= (\Lambda F \Lambda^T)^{\mu\nu}\end{aligned}$$

so with all these matrix definitions, we see $F' = \Lambda F \Lambda^T$. So if F is constructed from the quantities we measure in an inertial frame $\mathcal{S}$, and Λ is the Lorentz transformation relating this frame to another inertial frame $\mathcal{S}'$, then the elements of F' give the electric and magnetic fields in $\mathcal{S}'$.

To see how this works in a specific example, consider a boost in the positive x-direction. We know that the appropriate 4×4 matrix giving the Lorentz transformation is

$$\Lambda = \begin{pmatrix} \gamma(v) & -\frac{\gamma(v)v}{c} & 0 & 0 \\ -\frac{\gamma(v)v}{c} & \gamma(v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As we said, F is the upper-index version of the field strength tensor, namely,

$$F = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}.$$

Conputing $\Lambda F \Lambda^T$ is a bit on the tedious side, but we can do it, and the result is

$$
\begin{aligned}
F' &= \Lambda F \Lambda^T \\
&= \begin{pmatrix}
0 & E_x/c & \frac{\gamma(v)}{c}(E_y - vB_z) & \frac{\gamma(v)}{c}(E_z + vB_y) \\
-E_x/c & 0 & \gamma(v)(B_z - vE_y/c^2) & -\gamma(v)(B_y + vE_z/c^2) \\
-\frac{\gamma(v)}{c}(E_y - vB_z) & -\gamma(v)(B_z - vE_y/c^2) & 0 & B_x \\
-\frac{\gamma(v)}{c}(E_z + vB_y) & \gamma(v)(B_y + vE_z/c^2) & -B_x & 0
\end{pmatrix}.
\end{aligned}
$$

(Note this is antisymmetric, as expected.)

Now we can read off the transformed fields: the 01 component of F' is F'^{01}, which is E'_x/c, and thus we see $E'_x = E_x$: the component of the electric field in the direction of the boost is unchanged. Looking at the 23 component says the same is true for the magnetic field: $B'_x = B_x$. (This is already different from a Lorentz transformation, where it's the spacial components *perpendicular* to a boost which don't change.) If we look at the 21 component, this is $-B'_z$ and we see $B'_z = \gamma(v)(B_z - vE_y/c^2)$. And so on. So, writing everything out, we see the transformation rules for the electromagnetic field for a boost in the positive x-direction are

$$
\begin{aligned}
E'_x &= E_x, & B'_x &= B_x, \\
E'_y &= \gamma(v)\left(E_y - vB_z\right), & B'_y &= \gamma(v)\left(B_y + \frac{v}{c^2}E_z\right), \\
E'_z &= \gamma(v)\left(E_z + vB_y\right), & B'_z &= \gamma(v)\left(B_z - \frac{v}{c^2}E_y\right).
\end{aligned}
$$

Showing the *general* case is eminently doable but a bit lengthy, so we don't do it here and just state the result: if $\vec{E}$ and $\vec{B}$ are the fields in an inertial frame $\mathcal{S}$, and $\mathcal{S}'$ moves at a constant velocity $\vec{v}$ relative to $\mathcal{S}$, then the fields in $\mathcal{S}'$ are given by

$$
\begin{aligned}
E'_\parallel &= E_\parallel, & \vec{E}'_\perp &= \gamma(v)\left(\vec{E}_\perp + \vec{v} \times \vec{B}\right), \\
B'_\parallel &= B_\parallel, & \vec{B}'_\perp &= \gamma(v)\left(\vec{B}_\perp - \vec{v} \times \vec{E}\right),
\end{aligned}
$$

where a $\parallel$ subscript indicates the component in the same direction as $\vec{v}$ and $\perp$ denotes the two components perpendicular to the velocity. (So, for the example we did, the velocity is $\vec{v} = v\hat{e}_x$ and thus the x-component is the parallel one and the y- and z-components are the perpendicular ones.)

Also note that since crossing a field with $\vec{v}$ will eliminate the component parallel to $\vec{v}$, $\vec{v} \times \vec{E} = \vec{v} \times \vec{E}_\perp$ and $\vec{v} \times \vec{B} = \vec{v} \times \vec{B}_\perp$ and thus the transformations of the parallel and perpendicular components are entirely independent of each other.

These laws are useful in general, but they now can be used to show exactly *why* a moving charge creates a magnetic field, and we'll now do that. So suppose we start with a point charge which moves with a constant velocity in an inertial frame $\mathcal{S}$ (which we'll call the "lab frame" since it's where we're measuring our fields). Without loss of generality, define the positive x-direction of a Cartesian coordinate system to be the same as the velocity, so $\vec{v} = v\hat{e}_x$. We now ask, what is the electromagnetic field in this frame?

We could, naturally, compute $\vec{E}$ and $\vec{B}$ using all of the formulae we've derived over the semester, but we don't have to if we think of what's going on in the charge's *rest* frame. So let $\mathcal{S}'$ be the frame in which the charge is at rest; for this to be the case, the rest frame is moving with velocity $\vec{v}$ relative to the lab frame, and so we know the fields are related by

$$\begin{aligned} E'_x = E_x, &\qquad B'_x = B_x, \\ E'_y = \gamma(v)\left(E_y - vB_z\right), &\qquad B'_y = \gamma(v)\left(B_y + \frac{v}{c^2}E_z\right), \\ E'_z = \gamma(v)\left(E_z + vB_y\right), &\qquad B'_z = \gamma(v)\left(B_z - \frac{v}{c^2}E_y\right). \end{aligned}$$

However, from the rest frame's point of view, it sees the primed fields and says the lab frame is moving at velocity $-v\hat{e}_x$, so the inverse transformations are obtained simply by swapping the primed and unprimed quantities and changing the sign of v:

$$\begin{aligned} E_x = E'_x, &\qquad B_x = B'_x, \\ E_y = \gamma(v)\left(E'_y + vB'_z\right), &\qquad B_y = \gamma(v)\left(B'_y - \frac{v}{c^2}E'_z\right), \\ E_z = \gamma(v)\left(E'_z - vB'_y\right), &\qquad B_z = \gamma(v)\left(B'_z + \frac{v}{c^2}E'_y\right). \end{aligned}$$

(Remember that $\gamma(v)$ is an even function of v, so $\gamma(-v) = \gamma(v)$.) In the rest frame, we have only a single, unmoving particle at the origin, and since there are no moving changes, there is no magnetic field, so $B'_x = B'_y = B'_z = 0$. Furthermore, we also know the electric field from Coulomb's law, but we must be sure to express it in the *primed* spacetime coordinates:

$$\vec{E}'(t', \vec{r}') \quad = \quad \frac{q}{4\pi\epsilon_o} \frac{\vec{r}'}{|\vec{r}'|^3},$$

or

$$
\begin{aligned}
E'_x(t',x',y',x') &= \frac{q}{4\pi\epsilon_0}\frac{x'}{[(x')^2+(y')^2+(z')^2]^{3/2}},\\
E'_y(t',x',y',x') &= \frac{q}{4\pi\epsilon_0}\frac{y'}{[(x')^2+(y')^2+(z')^2]^{3/2}},\\
E'_z(t',x',y',x') &= \frac{q}{4\pi\epsilon_0}\frac{z'}{[(x')^2+(y')^2+(z')^2]^{3/2}}.
\end{aligned}
$$

We can now use the above inverse transformations to get the fields in the lab frame, but we also need to express the spacetime coordinates in this frame as well, i.e.

$$
t' = \gamma(v)\left(t - \frac{v}{c^2}x\right), \qquad x' = \gamma(v)\left(x - vt\right),
$$
$$
y' = y, \qquad z' = z
$$

and all this together gives the electric field components as

$$
\begin{aligned}
E_x(t,x,y,z) &= E'_x(t',x',y',z')\\
&= \frac{\gamma(v)q}{4\pi\epsilon_0}\frac{x-vt}{[\gamma^2(v)(x-vt)^2+y^2+z^2]^{3/2}},\\
E_y(t,x,y,z) &= \gamma(v)E'_y(t',x',y',z')\\
&= \frac{\gamma(v)q}{4\pi\epsilon_0}\frac{y}{[\gamma^2(v)(x-vt)^2+y^2+z^2]^{3/2}},\\
E_z(t,x,y,z) &= \gamma(v)E'_z(t',x',y',z')\\
&= \frac{\gamma(v)q}{4\pi\epsilon_0}\frac{z}{[\gamma^2(v)(x-vt)^2+y^2+z^2]^{3/2}}.
\end{aligned}
$$

The appearance of the $x - vt$ here makes perfect sense, since the particle's moving with speed v in the positive x-direction. However, it's worth noting that this is *not* radially symmetric. The $\gamma^2(v)$ multiplying the $(x-vt)^2$ in the denominator has the effect of "squashing" the electric field in the direction of motion: if we pick a t and centre a sphere around the charge, the magnitude of $\vec{E}$ declines as you move along the sphere's surface toward the two poles where the sphere intersects the z-axis.

Now, the magnetic field was zero in the rest frame, but it's decidedly *not* zero in the lab frame. Well, okay, the x-component is (because $B_x = B'_x = 0$), but the other two components are

$$\begin{aligned}
B_y(t,x,y,z) &= -\frac{\gamma(v)v}{c^2}\,E'_z(t',x',y',z') \\
&= -\frac{\mu_0\gamma(v)qv}{4\pi}\,\frac{z}{\left[\gamma^2(v)(x-vt)^2+y^2+z^2\right]^{3/2}}, \\
B_z(t,x,y,z) &= \frac{\gamma(v)v}{c^2}\,E'_y(t',x',y',z') \\
&= \frac{\mu_0\gamma(v)qv}{4\pi}\,\frac{y}{\left[\gamma^2(v)(x-vt)^2+y^2+z^2\right]^{3/2}}
\end{aligned}$$

(where we've used $c^2\epsilon_0 = 1/\mu_0$ for the umpteenth time). Notice that this makes $\vec{B}$ proportional to $-z\hat{e}_y + y\hat{e}_z$, which points *around* the x-axis (in the same way $\hat{e}_\phi$ points around the z-axis in cylindrical and spherical coordinates), and this is what we expect. If you take your right hand and point your thumb in the direction of the charge's motion, the magnetic fields wraps around this axis in the same directions as your finger do. And since a current is just a whole collection of moving point charges, this explains the direction of the magnetic field we find in the Biot-Savart law. So the creation of a magnetic field when you have a current is, in essence, a *result* of special relativity.

D. The Relativistic Form of Maxwell's Equations

Now that we have a way of incorporating the electromagnetic field into an inherently relativistic quantity (the field strength tensor), we can return to Maxwell's equations. Let's take the same approach as we did in finding the field strength tensor, by writing eveything in the equations in terms of 4-vectors and tensors.

We start with Gauss' law for electricity, $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$.Writing it out in full and using the 4-derivative, 4-current and field strength gives

$$\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \\
&= (\partial_1)\left(-cF^{10}\right) + (\partial_2)\left(-cF^{20}\right) + (\partial_3)\left(-cF^{30}\right) \\
&= \frac{J^0/c}{\epsilon_0}
\end{aligned}$$

or, using $c^2 = 1/\mu_0\epsilon_0$ yet again, $\partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = -\mu_0 J^0$. Note the left-hand side is summed only over the spacial coordinates, and therefore isn't automatically $\partial_\mu F^{\mu 0}$. However, recall that $F^{00} = 0$, so we can add $\partial_0 F^{00}$ to it without changing the right-hand side, in which case we end up with $\partial_\mu F^{\mu 0} = -\mu_0 J^0$.

Now let's look at the other equation with a source in it, Ampère's law. The x-component is

$$\begin{aligned}
\left(\vec{\nabla} \times \vec{B}\right)_x &= \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \\
&= (\partial_2)\left(-F^{21}\right) - (\partial_3)\left(F^{31}\right) \\
&= -(\partial_2 F^{21} + \partial_3 F^{31}) \\
&= \mu_0 J_x + \mu_0 \epsilon_0 \frac{\partial E_x}{\partial t} \\
&= \mu_0 (J^1) + \frac{1}{c^2}\,(c\partial_0)\left(cF^{01}\right) \\
&= \mu_0 J^1 + \partial_0 F^{01}.
\end{aligned}$$

If we put all the field strength parts on the same side, this may be rewritten as $\partial_0 F^{01} + \partial_2 F^{20} + \partial_3 F^{31} = -\mu_0 J^1$. If we do the same sort of stuff with the y- and z-components of Ampère's law, we find, respectively, $\partial_0 F^{02} + \partial_1 F^{12} + \partial_3 F^{32} = -\mu_0 J^2$ and $\partial_0 F^{03} + \partial_1 F^{13} + \partial_2 F^{23} = -\mu_0 J^3$. But since $F^{11} = 0$, we can add $\partial_1 F^{11}$ to the left-hand side of the x-component equation to get $\partial_0 F^{01} + \partial_1 F^{11} + \partial_2 F^{20} + \partial_3 F^{31} = \partial_\mu F^{\mu 1} = -\mu_0 J^1$. F^{22} and F^{33} are also zero, so adding $\partial_2 F^{22}$ and $\partial_3 F^{33}$ to, respectively, the left-hand sides ofthe y- and z-component equations will give $\partial_\mu F^{\mu 2} = -\mu_0 J^2$ and $\partial_\mu F^{\mu 3} = -\mu_0 J^3$.

And we have what we're after, because we see that these four equations may now be written in a single form:

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu.$$

Because the index μ is summed over, a Lorentz transformation only sees the free index ν on the left-hand side and so $\partial_\mu F^{\mu\nu}$ will transform as a 4-vector, and the above equation says that it's specifically the 4-vector $-\mu_0 J^\nu$. Thus, if we have this equation in a frame $\mathcal{S}$ and transform to a new frame $\mathcal{S}'$, this becomes $\partial'_\mu F'^{\mu\nu} = -\mu_0 J'^\nu$; the equation has the same form in terms of all the quantities you measure in $\mathcal{S}'$ as it did in terms of the quantities in $\mathcal{S}'$.

So this takes care of the Maxwell equations involving sources, but there are four others: Gauss' law for magnetism (a single scalar equation) and Faraday's law of induction (a three-component vector equation). The fact that there are four of these as well is promising, since we hope to get a 4-vector equation like we did above. So let's start with Gauss' law for magnetism:

$$\begin{aligned}
\vec{\nabla}\cdot\vec{B} &= \frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z} \\
&= \partial_1 F^{23}+\partial_2 F^{31}+\partial_3 F^{12} \\
&= 0.
\end{aligned}$$

This is different that what we had before; in all the source equations, the index on the derivative always showed up as one of the field strength indices as well (e.g. $\partial_1 F^{10}$). Here we see that none the indices are shared in any of the three terms. We see that same thing happens in Faraday's law: the x-component gives

$$\begin{aligned}
\left(\vec{\nabla}\times\vec{E}\right)_x &= \frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z} \\
&= \partial_2(cF^{03})-\partial_3(-cF^{20}) \\
&= c(\partial_2 F^{03}+\partial_3 F^{20}) \\
&= -\frac{\partial B_x}{\partial t} \\
&= -c\partial_0 F^{23}
\end{aligned}$$

or $\partial_0 F^{23}+\partial_2 F^{03}+\partial_3 F^{20}=0$. The y- and z-components will similarly give $\partial_0 F^{13}+\partial_1 F^{30}+\partial_3 F^{01}=0$ and $\partial_0 F^{12}+\partial_1 F^{02}+\partial_2 F^{10}=0$.

These equations may look utterly baffling, but the next step is somewhat clearer if we write eveything in terms of the field strength tensor with two *lowered* indices, e.g. $F^{23}=F_{23}$, $F^{30}=-F_{30}=F_{03}$ and so on. Doing this renders the four equations as

$$\begin{aligned}
\partial_1 F_{23}+\partial_2 F_{31}+\partial_3 F_{12} &= 0, \\
\partial_0 F_{23}+\partial_2 F_{32}+\partial_3 F_{02} &= 0, \\
\partial_0 F_{13}+\partial_1 F_{03}+\partial_3 F_{10} &= 0, \\
\partial_0 F_{12}+\partial_1 F_{20}+\partial_2 F_{01} &= 0.
\end{aligned}$$

Notice that there's no 0 index in the first equation, and each term contains a 1, 2 and 3 index. There's no 1 in the second equation, but 0, 2 and 3 appear. In the third, no 2 but a 0, 1 and 3 and finally, no 3 but a 0, 1 and 2.

We've seen something like this before: the x-component of $\vec{a}\times\vec{b}$ contains only y- and z-components, etc. And this is because of the Levi-Civita symbol:

$\sum_{j,k} \epsilon_{ijk} a_j b_k$ means that whatever value i has will be skipped in the sum because $\epsilon_{ijk} = 0$ if any two indices are the same. So is there something like a cross-product involved in these four equations we've derived?

Yes, there is. We now extend our definition of the usual Levi-Civita symbol to the 4-dimensional Levi-Civita tensor, defined very similarly:

$$\epsilon^{\mu\nu\lambda\rho} = \begin{cases} +1 & \text{if } (\mu\nu\lambda\rho) \text{ is an even permutation of } (0123) \\ -1 & \text{if } (\mu\nu\lambda\rho) \text{ is an odd permutation of } (0123) \\ 0 & \text{if any two indices are the same} \end{cases}$$

This is an *invariant* tensor: the four upper indices means that it transforms appropriately, namely,

$$\epsilon^{\mu\nu\lambda\rho} \mapsto \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\lambda{}_\gamma \Lambda^\rho{}_\delta \epsilon^{\alpha\beta\gamma\delta}.$$

The 4-dimensional ϵ has the same relation with 4×4 matrices as the 3-dimensional ϵ has with 3×3 matrices, namely, the above is $(\det \Lambda)\epsilon^{\mu\nu\lambda\rho}$. But all proper Lorentz transformations have determinant 1, so this means $\epsilon^{\mu\nu\lambda\rho}$ is the same in all inertial frames, which is why it can be consistently defined as above.

Where it comes into Maxwell's equations is through a new object, called the *dual field strength tensor*, denoted $\star F$ (some texts use $\tilde{F}$) and defined as

$$\star F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}.$$

Thus, its components are the components of F but in a different order. However, even with no further computation, we can see one property of $\star F$: it's antisymmetric, because swapping $\mu\nu$ in the Levi-Civita tensor changes its sign.

Now to compute one of its components, $\star F^{01}$: to find this, we need $\epsilon^{01\lambda\rho} F_{\lambda\rho}$, summed over all λ and ρ from 0 to 3. But since 0 and 1 already appear in the ϵ, we get a zero if λ or ρ takes on either of these values. Furthermore, if λ are ρ are the same, we'll get a zero. Thus, even though there are sixteen terms in the full sum $\epsilon^{01\lambda\rho} F_{\lambda\rho}$, only two survive, the ones where $(\lambda\rho)$ is (23) or (32): $\epsilon^{01\lambda\rho} F_{\lambda\rho} = \epsilon^{0123} F_{23} + \epsilon^{0132} F_{32}$. (0123) is quite obviously an even permutation of (0123) and since (0132) is obtained from (0123) by a single swap (of 2 and 3), it's an odd permutation. Furthermore, $F_{32} = -F_{23}$, so we see

$$\begin{aligned}\star F^{01} &= \frac{1}{2}\epsilon^{01\lambda\rho}F_{\lambda\rho} \\ &= \frac{1}{2}\left[(+1)(F_{23}) + (-1)(-F_{23})\right] \\ &= F_{23}.\end{aligned}$$

Similar computations give the other five independent components, and we get the following:

$$\begin{aligned}\star F^{01} = -\star F^{10} = &\ F_{23} = -F_{32} &= B_x, \\ \star F^{02} = -\star F^{20} = &\ F_{31} = -F_{13} &= B_y, \\ \star F^{03} = -\star F^{10} = &\ F_{12} = -F_{21} &= B_z, \\ \star F^{12} = -\star F^{21} = &\ F_{03} = -F_{30} &= -E_z/c, \\ \star F^{23} = -\star F^{32} = &\ F_{01} = -F_{10} &= -E_x/c, \\ \star F^{13} = -\star F^{31} = &\ F_{20} = -F_{02} &= E_y/c,\end{aligned}$$

and $\star F^{00} = \star F^{11} = \star F^{22} = \star F^{33} = 0$.

Like the field strength tensor, we can think of this as a 4×4 matrix, and written this way, we get

$$\star F^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}.$$

(We see that this matrix is $F^{\mu\nu}$ with the replacements $\vec{E} \to c\vec{B}$ and $\vec{B} \to -\vec{E}/c$; this is an example of what's called a *duality transformation*, which is why the term "dual field strength tensor" is used.)

To see why this is exactly the object we need, let's rewrite Gauss' law for magnetism using the dual field strength tensor

$$\begin{aligned}\vec{\nabla} \cdot \vec{B} &= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \\ &= \partial_1(-\star F^{10}) + \partial_2(-\star F^{20}) + \partial_3(-\star F^{30}) \\ &= 0\end{aligned}$$

and now we see a repeated index in each term. Since $\star F^{00} = 0$, we can add zero to the above in the form of $-\partial_0 \star F^{00}$, and we get $\partial_\mu \star F^{\mu 0} = 0$. And it will probably come as no surprise at this point that the other three are equivalent to $\partial_\mu \star F^{\mu 1} = 0$, $\partial_\mu \star F^{\mu 2} = 0$ and $\partial_\mu \star F^{\mu 3} = 0$, and thus we can finally write *all* of Maxwell's equations in their relativistic form:

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu, \qquad \partial_\mu \star F^{\mu\nu} = 0.$$

Let's now see what happens when we express these in terms of the potentials. Let's look at the one involving the dual field strength first: we see

$$\begin{aligned}
\partial_\mu \star F^{\mu\nu} &= \partial_\mu \left[\frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\left(\partial_\lambda A_\rho - \partial_\rho A_\lambda\right)\right] \\
&= \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\partial_\mu\partial_\lambda A_\rho - \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\partial_\mu\partial_\rho A_\lambda.
\end{aligned}$$

The Levi-Civita tensor is antisymmetric in all its indices, but $\partial_\mu\partial_\lambda$ and $\partial_\mu\partial_\rho$ are *symmetric* in their indices due to the commutation of partial derivatives. And whenever you sum a symmetric pair of indices with an antisymmetric pair, you get zero, and thus $\partial \star F^{\mu\nu} = 0$ is automatically satisfied. But this is just the relativistic version of the motivation for the potentials: by introducing them, the source-free Maxwell equations are immediately solved. (As a teaser for those of you who may go on to study higher-level differential geometry, this is an application of what's called the Poincaré lemma. The $\star$ I use for the dual field strength also comes from differential geometry as well.)

Now for the equation with the 4-current in it: putting in the potentials gives

$$\begin{aligned}
\partial_\mu F^{\mu\nu} &= \partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \\
&= \partial_\mu\partial^\mu A^\nu - \partial_\mu\partial^\nu A^\mu \\
&= -\mu_0 J^\nu.
\end{aligned}$$

$\partial_\mu\partial^\mu = \Box$, the d'Alembertian. And we can swap the order of the partial derivatives in the second term to get

$$\Box A^\nu - \partial^\nu\left(\partial_\mu A^\mu\right) = -\mu_0 J^\nu.$$

Notice the $\partial_\mu A^\mu$; this is a kind of "4-divergence", and in principle can be anything, making the above the most general differential equation giving the 4-potential. However, we know that gauge invariance allows us to pick potentials which have particularly convenient properties, and we can use it to require them to satisfy the Lorentz gauge, which as we showed a couple of lectures ago is $\partial_\mu A^\mu = 0$, which gives $\Box A^\nu = -\mu_0 J^\nu$, the form we used in finding our time-dependent fields.

E. The Lorentz Force Law

Recall that in the very first lecture, we stated precisely all the basic building blocks we need to study electromagnetism, and it seems that we've got everything: the continuity equation in the form $\partial_\mu J^\mu = 0$ and Maxwell's equations $\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu$ and $\partial_\mu \star F^{\mu\nu} = 0$. But in a very nice way of bookending the module, we now finish with the one law we haven't yet put in relativistic form, the Lorentz force law.

But unlike the other equations, which were already relativistically-covariant (although it wasn't obvious in the original vector calculus notation), the Lorentz force law isn't. In other words, the formula

$$m\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = q\vec{E} + q\vec{u} \times \vec{B}$$

isn't actually correct, but is the *nonrelativistic* approximation of the correct law. To see this isn't hard: we know experimentally that a magnetic field causes a charge to feel a force only if it has a nonzero velocity $\vec{u}$. In relativity, the velocity of a particle is described properly by the velocity 4-vector U^μ which has components $U^0 = \gamma(u)c$ and $U^{1,2,3} = \gamma(u)u_{x,y,z}$, where we must keep in mind that $\vec{u}$ is dependent on the spacetime coordinates of the inertial frame we're in. Thus, any relativistic formulation of the force law has to use the 4-vector U and not just the 3-velocity $\vec{u}$.

However, there's already a big hint of what we need to do: pick the x-component of the force law above and reexpress it in terms of the 4-velocity and field strength:

$$\begin{aligned}
m\frac{\mathrm{d}u_x}{\mathrm{d}t} &= m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{U^1}{\gamma(u)}\right) \\
&= qE_x + q(\vec{u} \times \vec{B})_x \\
&= q\left(\frac{E_x}{c}\,c + B_z u_y - B_y u_z\right) \\
&= q\left(-F^{10}\frac{U^0}{\gamma(u)} + F^{12}\frac{U^2}{\gamma(u)} + F^{13}\frac{U^3}{\gamma(u)}\right) \\
&= \frac{q}{\gamma(u)}\left(F^{10}U_0 + F^{11}U_1 + F^{12}U_2 + F^{13}U_3\right)
\end{aligned}$$

because $F^{11} = 0$. And we see a contracted index appearing rather nicely: $F^{10}U_0 + F^{11}U_1 + F^{12}U_2 + F^{13}U_3 = F^{1\nu}U_\nu$. The index 1 showing up for the x-component is also incredibly suggestive, and finding that the index is 2 for the y-component and 3 for the z-component only hammers the point home.

There are γ-factors here and these can only show up when we've chosen a specific reference frame and can't appear explicitly in any general 4-vector expression. However, in the nonrelativistic limit, γ is near 1 and so the x-component becomes

$$m\frac{\mathrm{d}U^1}{\mathrm{d}t} \approx qF^{1\nu}U_\nu$$

and similarly for the y- and z-components. This is starting to look like our other equations and we might be tempted to propose $m\mathrm{d}U^\mu/\mathrm{d}t = qF^{\mu\nu}U_\nu$ as the correct version of the force law, but this can't be correct because t is specific to our coordinate system: it's the time in our chosen reference frame, and thus would change in a different frame and the proposed equation wouldn't transform as a 4-vector.

So we need a Lorentz-invariant analogue of the time, and we of course have one, the proper time τ defined via $\mathrm{d}\tau^2 = -\mathrm{d}s^2/c^2$ (which is $\mathrm{d}t^2 - |\mathrm{d}\vec{r}|^2/c^2$ if we pick a reference frame). Again, if all velocities in our reference frame are much less than c, There's virtually no difference between τ and t and so $\mathrm{d}u_x/\mathrm{d}t \approx \mathrm{d}U^1/\mathrm{d}\tau$. Therefore, since it has the Lorentz force law we've been using as its nonrelativistic limit, we propose that

$$m\frac{\mathrm{d}U^\mu}{\mathrm{d}\tau} = qF^{\mu\nu}U_\nu$$

is the correct 4-vector version of the law, and it is indeed correct. Thus, any analysis of the kinematics of a charged particle in an EM field has to use this law rather than the nonrelativistic approximation if it's to accurately describe the particle's motion.

But this "Lorentz 4-force law" also incorporates something else that we didn't talk about: we motivated this by looking at a 3-vector equation, but we've proposed a 4-vector law: what's the time component of this telling us? If we pick $\mu = 0$ in the above we get

$$\frac{\mathrm{d}U^0}{\mathrm{d}\tau} = qF^{0\nu}U_\nu$$

or

$$\begin{aligned}
\frac{\mathrm{d}U^0}{\mathrm{d}\tau} &= m\frac{\mathrm{d}}{\mathrm{d}\tau}\gamma(u)c \\
&= q\left(F^{00}U_0 + F^{01}U_1 + F^{02}U_2 + F^{03}U_3\right) \\
&= \frac{q\gamma(u)}{c}\left(E_x u_x + E_y u_y + E_z u_z\right)
\end{aligned}$$

or, since $\mathcal{E} = m\gamma(u)c^2$ is the particle's energy, a bit of rearrangement gives

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\tau} = q\gamma(u)\vec{E}\cdot\vec{u}$$

which tells us how the EM field changes the energy of the charge, and it agrees entirely with what we'd expect in the nonrelativistic limit: we know that if a particle with velocity $\vec{u}$ moves within a force field $\vec{f}$ (we'll use a small f to avoid confusion with the field strength), the rate at which the field does work on the object is $\vec{f}\cdot\vec{u}$, and is thus the rate at which the object's energy changes. In the nonrelativistic limit, $\vec{f} \approx q(\vec{E} + \vec{u}\times\vec{B})$, and since the second term is normal to the velocity, $\vec{f}\cdot\vec{u} \approx q\vec{E}\cdot\vec{u}$, exactly the limit of the zeroth component of our Lorentz 4-force law.