MP465 – Advanced Electromagnetism

Lectures 21 & 22 (30 April 2020)

2. The Far-Zone Approximation

We finished the previous lecture talking about the near-zone approximation where $L \ll r \ll \lambda$. I did a quick calculation which showed that the power carried away by the EM wave falls as $1/r^3$ and thus would seem to indicate that as r increases, the power gets smaller and smaller.

But this may not actually be true, because as r increases, at some point it will cease to be much smaller than the wavelength λ and the near-zone approximation no longer applies. In fact, if we get far enough away, r will actually eventually become much *greater* than λ, and this region, where r is taken to be much larger than both L and λ, is the one where the *far-zone* aproximation holds.

So how does this change the situation? Recall that, under the assumption $r \gg L$, we found the leading order term in the magnetic field was

$$\tilde{\vec{B}} \approx \frac{\mu_0}{4\pi}(ikr-1)\left(\frac{i\omega\tilde{\vec{p}}_0 \times \vec{r}\,e^{ikr}}{r^3}\right).$$

The near-zone approximation is $kr \ll 1$ and so we took $ikr - 1 \approx -1$, but in the far-zone approximation, $kr \gg 1$ and so $ikr - 1 \approx ikr$, and thus the magnetic field is approximately

$$\begin{aligned}\tilde{\vec{B}} &\approx \frac{\mu_0}{4\pi}(ikr)\left(\frac{i\omega\tilde{\vec{p}}_0 \times \vec{r}\,e^{ikr}}{r^3}\right)\\ &= \frac{\mu_0\omega^2}{4\pi c}\left(\frac{\vec{r}\times\tilde{\vec{p}}_0}{r^2}\right)e^{ikr}.\end{aligned}$$

The electric field amplitude is obtained via

$$\tilde{\vec{E}} = \frac{ic^2}{\omega}\vec{\nabla}\times\tilde{\vec{B}}.$$

Note that

$$
\begin{aligned}
\vec{\nabla} \times \left[\left(\frac{\vec{r} \times \tilde{\vec{p}}_0}{r^2} \right) e^{ikr} \right] &= \left[\vec{\nabla} \times \left(\frac{\vec{r} \times \tilde{\vec{p}}_0}{r^2} \right) \right] e^{ikr} - \left(\frac{\vec{r} \times \tilde{\vec{p}}_0}{r^2} \right) \times \left(\vec{\nabla} e^{ikr} \right) \\
&= \left[\vec{\nabla} \times \left(\frac{\vec{r} \times \tilde{\vec{p}}_0}{r^2} \right) - \left(\frac{\vec{r} \times \tilde{\vec{p}}_0}{r^2} \right) \times ik\hat{e}_r \right] e^{ikr}.
\end{aligned}
$$

k is large in the far-zone approximation, so the second term above is much larger than the first, and so

$$
\begin{aligned}
\tilde{\vec{E}} &\approx \frac{ic^2}{\omega} \vec{\nabla} \times \left[\frac{\mu_0 \omega^2}{4\pi c} \left(\frac{\vec{r} \times \tilde{\vec{p}}_0}{r^2} \right) e^{ikr} \right] \\
&\approx \frac{ic^2}{\omega} \frac{\mu_0 \omega^2}{4\pi c} \left[- \left(\frac{\vec{r} \times \tilde{\vec{p}}_0}{r^2} \right) \times ik\hat{e}_r e^{ikr} \right] \\
&= \frac{c\tilde{\vec{B}} \times \vec{r}}{r}
\end{aligned}
$$

so if we put back in the time dependence, we find the actual real magnetic field in the far-zone approximation is

$$
\vec{B}(t, \vec{r}) \approx \mathrm{Re} \left[\frac{\mu_0 \omega^2}{4\pi c} \left(\frac{\vec{r} \times \tilde{\vec{p}}_0}{r^2} \right) e^{i(kr - \omega t)} \right]
$$

with the electric field given by $\vec{E}(t, \vec{r}) \approx c\vec{B}(t, \vec{r}) \times \hat{e}_r$.

Notice now that both fields go as $1/r$, so the Poynting vector will go as $1/r^2$. As we discussed before, the surface area element on a sphere of radius r goes as r^2, so the power that radiates though this element, $\vec{S} \cdot \mathrm{d}\vec{\sigma}$, is *imdependent* of r! In other words, no matter how large the sphere, a *nonzero* amount of power can be transmitted through it. This is the basic reason why we can communicate information via EM waves and explains why radio and television transmission (and related notions like radioastronomy) are actually possible.

Let's do a bit of quick numerical analysis: as I write this, I'm listening to the news on RTÉ Radio One at a frequency of 88.9 MHz. This gives a wavelength of $\lambda = 3.3$ metres. Now, that big huge antenna at RTÉ down in Dublin 4 is, say, $L \approx 50$ metres tall. (I tried looking up the actual height online, but couldn't find it, so that's purely a guess). We see from this that since $L > \lambda$, there actually is no near-zone approximation possible, but the far-zone applies provided r is much larger than 50 metres. Now a kilometre is only 20 times larger than this, but let's call that "much larger" and so we expect that the approximation we've just derived applies for anything over a kilometre away. And that's why I can hear Brian Jennings deliver the latest pandemic news while sitting at my table.

That's the basic idea, but let's do some mathematical calculations to get an actual expressions for the transmitted power. Now, using the same reasoning as we used when looking at plane waves, for rapidly oscillating systems the power we measure is actually the time-averaged power given by $\mathrm{d}\bar{P} = \langle\vec{S}\rangle \cdot \mathrm{d}\vec{\sigma}$. We know that if our electric and magnetic fields have the form

$$\vec{E} = \mathrm{Re}\left[\tilde{\vec{E}}e^{-i\omega t}\right], \qquad \vec{B} = \mathrm{Re}\left[\tilde{\vec{B}}e^{-i\omega t}\right]$$

then the time-averaged Poynting vector will be

$$\langle\vec{S}\rangle = \frac{1}{2\mu_0}\tilde{\vec{E}} \times \tilde{\vec{B}}^*.$$

We have both $\tilde{\vec{E}}(\vec{r})$ and $\tilde{\vec{B}}(\vec{r})$ in the far-zone approximation, so the time-averaged Poynting vector will be

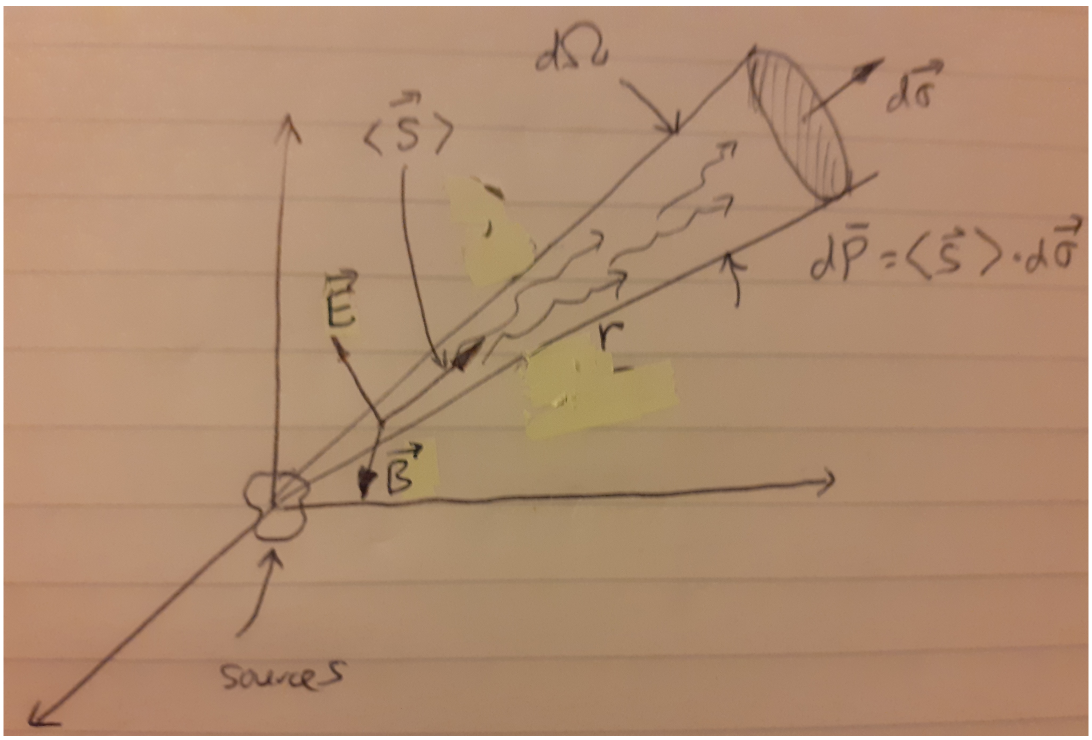
$$\begin{aligned}
\langle\vec{S}\rangle &\approx \frac{1}{2\mu_0}\left(\frac{c\tilde{\vec{B}}(\vec{r}) \times \vec{r}}{r}\right) \times \tilde{\vec{B}}^*(\vec{r}) \\
&= \frac{c}{2\mu_0 r}\left[\left(\tilde{\vec{B}}(\vec{r}) \cdot \tilde{\vec{B}}^*(\vec{r})\right)\vec{r} - \left(\vec{r} \cdot \tilde{\vec{B}}^*(\vec{r})\right)\tilde{\vec{B}}(\vec{r})\right].
\end{aligned}$$

But since $\tilde{\vec{B}}^* \propto \vec{r} \times \tilde{\vec{p}}_0^*$, it's perpendicular to $\vec{r}$, and so we find

$$\begin{aligned}
\langle\vec{S}\rangle &\approx \frac{c}{2\mu_0 r}|\tilde{\vec{B}}(\vec{r})|^2\vec{r} \\
&= \frac{c}{2\mu_0}\left|\frac{\mu_0\omega^2}{4\pi c}\left(\frac{\vec{r} \times \tilde{\vec{p}}_0}{r^2}\right)e^{ikr}\right|^2 \hat{e}_r \\
&= \frac{\mu_0\omega^4}{32\pi^2 c}\frac{|\hat{e}_r \times \tilde{\vec{p}}_0|^2}{r^2}\,\hat{e}_r.
\end{aligned}$$

(Remember that if $\vec{a}$ is a complex vector, $|\vec{a}|^2 = \vec{a} \cdot \vec{a}^*$.) Unsurprisingly, this points radially outward: energy is being transmitted directly away in all directions from the oscillating sources.

The surface area element on a sphere of radius r is $\mathrm{d}\vec{\sigma} = r^2 \mathrm{d}\Omega \, \hat{e}_r$, where $\mathrm{d}\Omega = \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$ is the solid angle subtended by this area element, as shown below:



Thus, the time-averaged radiated power is

$$\begin{aligned} \mathrm{d}\bar{P} &= \langle \vec{S} \rangle \cdot \mathrm{d}\vec{\sigma} \\ &= \frac{\mu_0 \omega^4}{32\pi^2 c} |\hat{e}_r \times \tilde{\vec{p}}_0|^2 \mathrm{d}\Omega \end{aligned}$$

which we see is indeed independent of r, and so remains nonzero no matter how far away from the sources we get. From this we define the time-averaged power distribution function to be the radiated power per solid angle, i.e.

$$\begin{aligned} \frac{\mathrm{d}\bar{P}}{\mathrm{d}\Omega} &= \langle \vec{S} \rangle \cdot \mathrm{d}\vec{\sigma} \\ &= \frac{\mu_0 \omega^4}{32\pi^2 c} |\hat{e}_r \times \tilde{\vec{p}}_0|^2 \end{aligned}$$

which will, in general, depend on the angular coordinates θ and ϕ. We would therefore get the total radiated power by integrating this over the entire surface of the sphere, namely,

$$
\begin{aligned}
\bar{P} &= \int_{4\pi} \frac{\mathrm{d}\bar{P}}{\mathrm{d}\Omega}\,\mathrm{d}\Omega \\
&= \int \frac{\mathrm{d}\bar{P}}{\mathrm{d}\Omega}(\theta,\phi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi
\end{aligned}
$$

where the integral is over $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$.

Let's do an example of how to calculate all of this for a given oscillating source: suppose we have a vertical wire of height L through which we run a current $I_0 \cos\omega t$, where I_0 is the peak current. If we pick a coordinate system such that the antenna lies along the z-axis with its lower end at the origin, then the current density is given in Cartesian coordinates by $\vec{J}(t,\vec{r}) = I_0 \cos\omega t\,\delta(x)\delta(y)\,\hat{e}_z$ for $0 \le z \le L$ and zero otherwise.

What we want is the complex dipole amplitude $\tilde{\vec{p}}_0$, because once we have that, we can compute eveything we need. The first step is to find the $\tilde{\vec{J}}(\vec{r})$ such that $\vec{J} = \mathrm{Re}[\tilde{\vec{J}}e^{-i\omega t}]$. If $\tilde{\vec{J}} = \vec{J}_R + i\vec{J}_I$ then $\vec{J} = \vec{J}_R\cos\omega t + \vec{J}_I \sin\omega t$, so we see quickly that $\vec{J}_R = I_0\delta(x)\delta(y)\hat{e}_z$ and $\vec{J}_I = \vec{0}$ and therefore $\tilde{\vec{J}}(x,y,z) = I_0\delta(x)\delta(y)\hat{e}_z$ for $0 \le z \le L$ and zero otherwise.

We derived a formula for getting the dipole amplitude from this:

$$
\begin{aligned}
\tilde{\vec{p}}_0 &= \frac{i}{\omega}\int \tilde{\vec{J}}(\vec{r})\,\mathrm{d}^3\vec{r} \\
&= \frac{iI_0\hat{e}_z}{\omega}\int \delta(x)\delta(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \frac{iI_0 L}{\omega}\,\hat{e}_z
\end{aligned}
$$

because the two delta-functions integrate to 1 and the z-integral is from 0 to L.

To get the power distribution, we need $|\hat{e}_r \times \tilde{\vec{p}}_0|^2$, which is

$$
\begin{aligned}
\left|\hat{e}_r \times \tilde{\vec{p}}_0\right|^2 &= \left|\hat{e}_r \times \left(\frac{iI_0L}{\omega}\,\hat{e}_z\right)\right|^2 \\
&= \frac{I_0^2L^2}{\omega^2}\,|\hat{e}_r \times \hat{e}_z|^2 .
\end{aligned}
$$

Recall that the magnitude of the cross-product of two vectors is the product of the magnitudes of the vectors times the sine of the angle between them. $\hat{e}_r$ and $\hat{e}_z$ both have unit magnitude, and the angle between them is, by definition, the spherical coordinate θ. Thus,

$$\left|\hat{e}_r \times \tilde{\vec{p}}_0\right|^2 \;=\; \frac{I_0^2 L^2}{\omega^2} \sin^2\theta.$$

Putting this into the formula for the power distribution gives

$$\frac{\mathrm{d}\bar{P}}{\mathrm{d}\Omega} \;=\; \frac{\mu_0 I_0^2 L^2 \omega^2}{32\pi^2 c} \sin^2\theta.$$

Now, at what values of θ is this maximised? It's zero at $\theta = 0$ and $\theta = \pi$, so very little power is radiated straight up or straight down. $\sin^2\theta = 1$ at $\theta = \pi/2$, so most of the power is transmitted horizontally, and this explains why radio and TV antennae are oriented the way they are: since the majority of receivers (home radios and TVs) are at ground level, you want the signal to be strongest there. Since this is exactly $\theta = \pi/2$, the transmitted power from the above (admittedly very simple) vertically-oriented antenna will be at its maximum precisely where all the listeners and viewers are.

The *total* power transmitted will be the integral of the above over all solid angles, i.e.

$$\begin{aligned}
\bar{P} &= \int \frac{\mathrm{d}\bar{P}}{\mathrm{d}\Omega} \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{\mu_0 I_0^2 L^2 \omega^2}{32\pi^2 c} \left(\int_0^\pi \sin^3\theta\,\mathrm{d}\theta\right) \left(\int_0^{2\pi} \mathrm{d}\phi\right) \\
&= \frac{\mu_0 I_0^2 L^2 \omega^2}{12\pi c}
\end{aligned}$$

so if you were a pirate radio DJ who had a limited amount of transmission power at your disposal, this formula would give you a rough idea of what range of frequencies, antenna heights and antenna currents would be available to you.

Now, virtually everything we've done in this section on time-dependent sources and fields has assumed a very specific time-dependence, namely, simple harmonic: that's where the $e^{-i\omega t}$ comes from. Although this assumption seems restrictive, it's actually not, and here's why: keep in mind that any time-dependent function $f(t)$ may be written as

$$f(t) \;=\; \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t} \mathrm{d}\omega$$

where (up to some numerical factor) $\tilde{f}(\omega)$ is the Fourier transform (FT) of $f(t)$. Thus, if we interpret all of our parameters written with a tilde as the FTs of the real quantity (i.e. the quantities without tildes), then all the expressions we've computed still hold. Keep in mind, though, that under this interpretation, the tilded quantities are functions of the frequency as well. For example, our far-zone approximation for the magnetic field still holds, but the proper expression is now $\vec{B}(t,\vec{r}) = \int_{-\infty}^{\infty} \tilde{\vec{B}}(\omega,\vec{r}) e^{-i\omega t}\mathrm{d}\omega$ with $\tilde{\vec{B}}(\omega,\vec{r})$ being the expression we derived, but with $\tilde{\vec{p}}_0$ replaced by the FT of $\vec{p}(t)$.

But this gives us a way of reverse-engineering the correct solution: under the FT interpretation, any factor of ω can only come from i times a time-derivative: for example, for the far-zone electric field, the $\omega^2\tilde{\vec{p}}_0$ can only come from $-\ddot{\vec{p}}(t)$. Thus, the expression we found,

$$\tilde{\vec{B}}(\vec{r}) \;\approx\; \frac{\mu_0\omega^2}{4\pi c}\left(\frac{\vec{r}\times\tilde{\vec{p}}_0}{r^2}\right) e^{ikr}$$

which gives

$$\vec{B}(t,\vec{r}) \;\approx\; \mathrm{Re}\left[\frac{\mu_0\omega^2}{4\pi c}\left(\frac{\vec{r}\times\tilde{\vec{p}}_0}{r^2}\right) e^{i(kr-\omega t)}\right]$$

becomes

$$\begin{aligned}
\vec{B}(t,\vec{r}) \;&\approx\; \int_{-\infty}^{\infty} \frac{\mu_0\omega^2}{4\pi c}\left(\frac{\vec{r}\times\tilde{\vec{p}}(\omega)}{r^2}\right) e^{i(kr-\omega t)}\mathrm{d}\omega \\
&= \frac{\mu_0}{4\pi c}\,\frac{\vec{r}}{r^2}\times\int_{-\infty}^{\infty}\omega^2\tilde{\vec{p}}(\omega)e^{-i\omega(t-r/c)}\mathrm{d}\omega \\
&= \frac{\mu_0}{4\pi c}\,\frac{\ddot{\vec{p}}(t-r/c)\times\hat{e}_r}{r}
\end{aligned}$$

which is the correct far-zone magnetic field for *any* type of time-dependence. (Notice the retarded time $t-r/c$ appearing in this expression!) Similar agruments and formulae hold for the other quantities we've derived, so the assumption of single-frequency oscillations is less specific than it seems. Which is great.

V. Relativistic Formulation of Electromagnetism

A. Quick Review of Special Relativity Basics

In our derivation of the Green's function for the d'Alembertian, we needed to invoke the notion of causality to get a physically-meaningful solution. We saw explicitly that the correct function imposed the restriction that information about the state of the sources had to come from the past and travelled at the speed of light. This hints at a deep connection between Maxwell's equations and special relativity, and in this final section of the module, that's exactly what we'll be looking at.

In fact, the very foundations were laid in a rather oblique way by the structure of Maxwell's equations. When Hendrik Lorentz came up with his famous tarnsformations, he wasn't thinking of inertial frames or the equivalence principle. He came up with them because he was looking for the most general set of linear transformations that left Maxwell's equations invariant. The fact that these selfsame transformations explain how 4-vectors change between inertial frames only came later. But this invariance says that we should be able to reformulate electromagnetism in terms of special relativity, so that's what we'll start doing right now.

Now, one of the prerequisites for this module is a solid background in special relativity, the equivalent of MP352, so I'm going to assume that everyone knows the fundamentals of Lorentz transformations, 4-vector notation, Einstein summation convention and the like. However, we'll do a quick review of the basics and notation now.

The first thing is to specify which metric we're using. If $\mathrm{d}x^\mu$ is a small change in the spacetime coordintes, then the resultant invariant length element due to this change is

$$\begin{aligned} \mathrm{d}s^2 &= -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ &= \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \end{aligned}$$

where $\eta_{00} = -1$, $\eta_{11} = \eta_{22} = \eta_{33} = 1$ and all other components are zero. Alternatively, we can think of $\eta_{\mu\nu}$ – called the "metric tensor", or simply the "metric" – as the $\mu^{\text{th}}\nu^{\text{th}}$-element of the matrix

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We denote by $\eta^{\mu\nu}$ the $\mu^{\text{th}}\nu^{\text{th}}$-element of the matrix η^{-1}, which happens to be the same as η (but those who've had a bit of general relativity know that, for a curved spacetime, the metric is usually *not* the same as its inverse).

The set of linear transformations that leave this invariant length element, well, invariant, are called Lorentz transformations, and can be defined as the set of 4×4 matrices Λ which satisfy the relation $\Lambda^T \eta \Lambda = \eta$. Alternatively, if $\Lambda^\mu{}_\nu$ is the $\mu^{\text{th}}\nu^{\text{th}}$-element of Λ, the defining relation becomes $\eta_{\alpha\beta}\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu = \eta_{\mu\nu}$.

These transformations can be one of two types: 3-dimensional rotations, which leave the zeroth-component of a 4-vector unchanged, and boosts, which give the relationship between quantities in different inertial (i.e. nonaccelerating) frames of reference moving at a constant velocity relative to one another. For example, suppose a frame $\mathcal{S}'$ is moving with a constant speed v in the $+x$-direction relative to a frame $\mathcal{S}$ (a "boost in the positive x-direction"). If w^μ is a 4-vector in $\mathcal{S}$, then its components in $\mathcal{S}'$ are

$$w'^0 = \gamma(v)\left(w^0 - \frac{v}{c}w^1\right), \qquad w'^1 = \gamma(v)\left(w^1 - \frac{v}{c}w^0\right),$$
$$w'^2 = w^2, \qquad w'^3 = w^3$$

where $\gamma(v) = (1 - v^2/c^2)^{-1/2}$. In matrix notation, this would be $w' = \Lambda \cdot w$ where $w^T = (w^0, w^1, w^2, w^3)$ and

$$\Lambda = \begin{pmatrix} \gamma(v) & -\frac{\gamma(v)v}{c} & 0 & 0 \\ -\frac{\gamma(v)v}{c} & \gamma(v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

or component-wise, $w'^\mu = \Lambda^\mu{}_\nu w^\nu$. However, this is how a *covariant* 4-vector changes, i.e. one with an upper index. We can also have *contravariant* 4-vectors, written with a lower index, and these transform according to the rule $w'_\mu = (\Lambda^{-1})\nu_\mu w_\nu$. This means that any quantity of the form $w^\mu u_\mu = w$ (the index μ is "contracted") is a Lorentz-invariant quantity. But the two types of 4-vectors are related, because you can always use the metric to transform one into the other type: $w_\mu = \eta_{\mu\nu} w^\nu$ and $w^\mu = \eta^{\mu\nu} w_\nu$. This allows us to define the norm-squared of a 4-vector w^μ as

$$\begin{aligned} w^2 &= \eta_{\mu\nu} w^\mu w^\nu = \eta^{\mu\nu} w_\mu w_\nu \\ &= w_\mu w^\mu = w^\mu w_\mu \\ &= -(w^0)^2 + (w^1)^2 + (w^2)^2 + (w^3)^2 = -(w_0)^2 + (w_1)^2 + (w_2)^2 + (w_3)^2. \end{aligned}$$

This can be of any sign: if it's postive, we call the 4-vector "spacelike", if it's negative, it's "timelike" and if it's zero, it's "null" or "lightlike".

But an object can have more than one index, and where the indices are located indicates how it transforms between inetrial frames: for example, if we denote a quantity in S by $A^{\mu\nu}$, this means that if Λ is a Lorentz ransformation taking us to the frame $\mathcal{S}'$, then this quantity's value in the primed frame is

$$A'^{\mu\nu} \;=\; \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}A^{\alpha\beta}.$$

Similarly, any object written as $B^{\mu}{}_{\nu}$ will transform according to the rule

$$B'^{\mu}{}_{\nu} \;=\; \Lambda^{\mu}{}_{\alpha}(\Lambda^{-1})^{\beta}{}_{\nu}B^{\alpha}{}_{\beta}$$

and so on.

B. The Current and Potential 4-Vectors

What quantities do we know are 4=vectors? In other words, what sets of four quantities obey the transformation laws we've just discussed? There are three that you hopefully all know: the coordinate 4-vector x^μ, the velocity 4-vector U^μ and the momentum 4-vector P^μ. In other words, the three column vectors

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix},\; U^\mu = \begin{pmatrix} \gamma(u)c \\ \gamma(u)u_x \\ \gamma(u)u_y \\ \gamma(u)u_z \end{pmatrix},\; P^\mu = \begin{pmatrix} m\gamma(u)c \\ m\gamma(u)u_x \\ m\gamma(u)u_y \\ m\gamma(u)u_z \end{pmatrix}$$

all transform between inertial frames $\mathcal{S}$ and $\mathcal{S}'$ via $w^\mu \mapsto w'^\mu = \Lambda^\mu{}_\nu w^\nu$. (In fact, the very definition of a covariant 4-vector is "anything that transforms the same way the spacetime coordinates do".) So, for example, if a particle has momentum $\vec{p}$ and energy $E = \sqrt{|\vec{p}|^2c^2 + m^2c^4}$ in a frame $\mathcal{S}$, then if $\mathcal{S}'$ is moving with velocity $v\hat{e}_x$ relative to $\mathcal{S}$, then the particle's energy and momentum in the primed frame is

$$E' = \gamma(v)\left(E - vp_x\right), \qquad p'_x = \gamma(v)\left(p_x - \frac{v}{c^2}E\right),$$
$$p'_y = p_y, \qquad p'_z = p_z$$

because $P^0 = E/c$ and $P^{1,2,3} = p_{x,y,z}$ are the components of a covariant 4-vector.

But there's also a paradigmatic contravariant 4-vector, namely, the space-time derivative $\partial/\partial x^\mu$, universally shorthanded to ∂_μ. So if we have a boost in the positive x-direction, the transformations of t, x, y and z are known and a simple application of the chain rule will show that $\partial'_\mu = (\Lambda^{-1})^\nu{}_\mu \partial_\nu$ and thus is a contravariant 4-vector. But this fact immediately begins to hint at a connection between relativity and electromagnetism: we saw that the d'Alembertian appeared in our equations for the potentials, and it's a *Lorentz-invariant* quantity. We can see this explicitly:

$$\begin{aligned}
\Box &= \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \\
&= -\left(\frac{\partial}{\partial(ct)}\right)^2 + \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2 \\
&= -(\partial_0)^2 + (\partial_1)^2 + (\partial_2)^2 + (\partial_3)^2 \\
&= \partial_\mu \partial^\mu
\end{aligned}$$

and because it's made of a contravariant and covariant vector contracted together, it's the same in all inertial frames: $\Box' = \Box$.

But there are also *sources* in Maxwell's equations, and these lead to another 4-vector. We'll show this for a simple 1-dimensional charge/current distribution, but the result can be generalised for arbitrary configuration. Consider a infinitely-long straight wire, at rest in an inertial frams $\mathcal{S}$, with a uniform linear charge density λ and a constant current I. Now, suppose we go into a frame $\mathcal{S}'$ which moves at a constant speed v in the same direction as the wire's length (call this the positive x-direction); what are the density λ' and current I' in this frame?

Pick a point on the wire, say, the origin. If we take a time inteval $(t, t+\Delta t)$ in $\mathcal{S}$, the amount of charge which flows through this point is $\Delta q = I\Delta t$. Now, look in the primed frame, where the interval is now $(t', t'+\Delta t')$: the wire is *not* at rest in this frame, it's moving at speed v in the negative x-direction. Thus, the total charge moving through it is not just Δq in the positive direction, but an additional charge $\lambda' v\Delta t'$ (since $v\Delta t'$ is the length of the wire segment which passes through the origin) moving in the negative direction; thus, the total charge moving thriough the origin in this frame is $\Delta q' = I\Delta t - \lambda' v \Delta t'$, and this defines the primed current $\Delta q' = I'\Delta t'$. Thus, $(I' + v\lambda')\Delta t' = I\Delta t$. But time dilation gives $\Delta t' = \gamma(v)\Delta t$, so we see that $I = \gamma(v)(I' + v\lambda')$.

This was picking a point in space but an interval in time; if we instead pick a point in time and an interval in space, a similar argument comparing the amount of charge between x and $x + \Delta x$ at a time t in $\mathcal{S}$ to the amount between x' and $x' + \Delta x'$ at a time t' in $\mathcal{S}'$ leads to , after invoking the length contraction $\Delta x' = \Delta x/\gamma(v)$, $\lambda = \gamma(v)(\lambda' + vI'/c^2)$. But we want the primed variables in terms of the unprimed ones, and it's a simple matter to show that these are

$$\lambda' = \gamma(v)\left(\lambda - \frac{v}{c^2}I\right), \qquad I' = \gamma(v)\left(I - v\lambda\right).$$

But note that this means that the two quantities λc and I transform *exactly* as the zeroth and first components of a 4-vector under a positive x-boost.

This was a simple 1-dimensional system, but it gives the flavour of the general result: for any system which has a charge density ρ and a current density $\vec{J}$ in a frame $\mathcal{S}$, then if $\mathcal{S}'$ is a frame boosted in the positive x-direction, the densities in this frame are

$$\rho' = \gamma(v)\left(\rho - \frac{v}{c^2}J_x\right), \qquad J'_x = \gamma(v)\left(J_x - v\rho\right),$$
$$J'_y = J_y, \qquad J'_z = J_z$$

which means that the four quantities $J^0 = \rho c$, $J^1 = J_x$, $J^2 - J_y$ and $J^3 = J_z$ are the components of the *current 4-vector* J^μ (also called the 4-current). And this leads to another extremely nice result: note that

$$\begin{aligned}
\partial_\mu J^\mu &= \partial_0 J^0 + \partial_1 J^1 + \partial_2 J^2 + \partial_3 J^3 \\
&= \left(\frac{1}{c}\frac{\partial}{\partial t}\right)\rho c + \left(\frac{\partial}{\partial x}\right)J_x + \left(\frac{\partial}{\partial y}\right)J_y + \left(\frac{\partial}{\partial z}\right)J_z \\
&= \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J}
\end{aligned}$$

so the continuity equation may be rewritten as $\partial_\mu J^\mu = 0$. This is an *extremely* important result: since it's a contraction of two 4-vectors, $\partial_\mu J^\mu$ is Lorentz-invariant. Thus, if it's zero in one inertial frame, it's zero in all inertial frames. But the continuity equation is an expression of conservation of charge, so this indicates that *total electric charge is Lorentz-invariant*; a system that has a total charge q in one frame has the same total charge in all frames. This is why electric charge is one of the quantities we can assign to fundamental particles; it's independent of the inertial frame we choose to measure it in. The charge of an electron is $-e$ whether it's at rest or moving at 99.9999999% of the speed of light.

But wait, there's more! Recall if we bring in the scalar and vector potentials, Maxwell's equantions lead to

$$\Box\Phi = -\frac{\rho}{\epsilon_0}, \qquad \Box\vec{A} = -\mu_o\vec{J}$$

provided the potentials satisfy the Lorentz (note the name!) gauge condition

$$\frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \vec{\nabla}\cdot\vec{A} \;\;=\;\; 0.$$

Using $\epsilon_0 = 1/\mu_0 c^2$, we can rewrite the above Maxwell equations in column-vector form as

$$\Box\begin{pmatrix}\Phi/c\\ A_x\\ A_y\\ A_z\end{pmatrix} \;\;=\;\; -\mu_0\begin{pmatrix}\rho c\\ J_x\\ J_y\\ J_z\end{pmatrix}.$$

The right-hand side transforms as a 4-vector (the constant factor $-\mu_0$ doesn't change that). The d'Alembertian doesn't change at all under a Lorentz transformation. Thus, the column vector containing the potentials *must be a 4-vector*. This is the potential 4-vector, or 4-potential, A^μ, and its components are $A^0 = \Phi/c$, $A^1 = A_x$, $A^2 = A_y$ and $A^3 = A_z$. Therefore, the equations which give the potentials in terms of the sources is the 4-vector equation

$$\Box A^\mu \;\;=\;\; -\mu_0 J^\mu,$$

Now, the quantity $\partial_\mu A^\mu$ is a Lorentz-invariant quantity. But it's

$$\begin{aligned}\partial_\mu A^\mu &= \left(\frac{1}{c}\frac{\partial}{\partial t}\right)\frac{\Phi}{c} + \left(\frac{\partial}{\partial x}\right)A_x + \left(\frac{\partial}{\partial y}\right)A_y + \left(\frac{\partial}{\partial z}\right)A_z\\ &= \frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \vec{\nabla}\cdot\vec{A}.\end{aligned}$$

Note that this isn't *automatically* zero. That's a particular choice for our potentials. But the above shows that the Lorentz gauge $\partial_\mu A^\mu = 0$ is (duh) Lorentz-invariant; if the potentials satisfy it in one inertial frame, they satisfy it in all inertial frames. This is in contrast to the Coloumb gauge $\vec{\nabla}\cdot\vec{A} = 0$, which is *not* Lorentz-invariant, so the Lorentz gauge is the most favoured gauge choice in situations where relativity plays a significant role (like, for example, particle physics).

Fine. But the potentials aren't the physical fields, the electric and magnetic fields are. How do they fit into this relativistic formulation? We can see by a simple counting argument that something different has to happen with them: we always need to specify four sources (one scalar density and one three-component vector density), and we get from them four potentials (one scalar and one vector). As we've shown, these collect themselves very nicely into two 4-vectors. But the two 3-vectors $\vec{E}$ and $\vec{B}$ contain *six* quantities. Not a multiple of four, and so it's not immediately obvious how we could get something like a 4-vector out of these.

And as it turns out, it's not a 4-vector that we need, but something called a 4-*tensor*, and that'll be the subject of the next lecture.