MP465 – Advanced Electromagnetism

Lectures 19 & 20 (23 April 2020)

IV. Electromagnetic Radiation

A. Solutions to the Time-Dependent Maxwell Equations

Right, back to the basics: Maxwell's equations in vacuum are, of course,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \qquad \vec{\nabla} \cdot \vec{B} = 0,$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

and the continuity equation is

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0.$$

(If we're in a linear medium, interpret the sources as the free sources and the permittivity and permeability as those of the medium.) In the case of a static system, where there is no time-dependence, then $\vec{E} = -\vec{\nabla}\Phi$ and $\vec{B} = \vec{\nabla} \times \vec{A}$ where the potentials satisfy

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}, \qquad \nabla^2 \vec{A} = -\mu_0 \vec{J}$$

with the choice that the vector potential satisfies the Coloumb gauge $\vec{\nabla} \cdot \vec{A} = 0$.

But suppose we *don't* assume time-independence. This is, of course, the more logical choice; virtually nothing in the universe is actually static. In this case, we need to keep all time derivatives, which means that even though $\vec{B} = \vec{\nabla} \times \vec{A}$ still holds, the electric field includes a time derivative of the vector potential: $\vec{E} = -\vec{\nabla}\Phi - \partial \vec{A}/\partial t$. These automatically satisfy Gauss' law for magnetism and Faraday's law of induction (the two source-free Maxwell equations), but Gauss' law for electricity now becomes

$$\begin{aligned}\vec{\nabla} \cdot \left(-\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \right) &= -\nabla^2 \Phi - \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) \\ &= \frac{\rho}{\epsilon_0}.\end{aligned}$$

Now, we're going to add zero to this and rearrange it:

$$\begin{aligned}
-\nabla^2\Phi - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) &= -\nabla^2\Phi - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) + \frac{1}{c^2}\left(\frac{\partial^2\Phi}{\partial t^2} - \frac{\partial^2\Phi}{\partial t^2}\right) \\
&= \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi - \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right) \\
&= \frac{\rho}{\epsilon_0}.
\end{aligned}$$

Notice the appearance of the d'Alembertian operator

$$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

acting on the scalar potential; using this notation, we may rewrite the above equation as

$$\Box\Phi = -\frac{\rho}{\epsilon_0} - \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right).$$

Now let's look at Ampère's law: putting in the potentials gives

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}\right)$$

which, after using $\mu_0\epsilon_0 = 1/c^2$ and $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$ gives

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} - \frac{1}{c^2}\left(\vec{\nabla}\frac{\partial\Phi}{\partial t} + \frac{\partial^2\vec{A}}{\partial t^2}\right)$$

and a bit of rearranging results in

$$\begin{aligned}
\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} &= \Box\vec{A} \\
&= -\mu_0\vec{J} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right).
\end{aligned}$$

Notice that the quantity in the brackets in this equation is the same as the one in the equation for Φ. Now, recall that potentials are not uniquely defined; if Φ and $\vec{A}$ give specific electric and magnetic fields, then for any function χ, $\Phi - \partial\chi/\partial t$ and $\vec{A} + \vec{\nabla}\chi$ give *exactly* the same fields. Thus, we can use this flexibility in defining our potentials to require them to satisfy the condition

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} \quad = \quad 0.$$

When we impose a condition on our potentials, it's called a gauge choice. In the static case, we chose the Coloumb gauge $\vec{\nabla} \cdot \vec{A} = 0$, but here we make the above choice, called (for reasons we'll see in a few lectures) the *Lorentz* gauge. Why this choice? Well, it simplifies the equations which the potentials must satisfy to

$$\Box\Phi = -\frac{\rho}{\epsilon_0}, \qquad \Box\vec{A} = -\mu_0\vec{J}.$$

Notice these are decoupled; the first says that the scalar potential is determined only by the charge density and the second that the vector potential is determined only by the current density.

How do we solve these? This is precisely what we discussed in this week's tutorial – we use the Green's function for the d'Alembertian. If we have a function satisfying

$$\Box_{t,\vec{r}}\, G(t, \vec{r}; t', \vec{r}') \quad = \quad \delta(t - t')\delta(\vec{r} - \vec{r}')$$

then the general solutions to the above two equations are

$$\begin{aligned} \Phi(t, \vec{r}) &= \Phi_0(t, \vec{r}) - \frac{1}{\epsilon_0}\int G(t, \vec{r}; t', \vec{r}')\rho(t', \vec{r}')\, \mathrm{d}t'\, \mathrm{d}^3\vec{r}', \\ \vec{A}(t, \vec{r}) &= \vec{A}_0(t, \vec{r})_0 - \mu_0 \int G(t, \vec{r}; t', \vec{r}')\vec{J}(t', \vec{r}')\, \mathrm{d}t'\, \mathrm{d}^3\vec{r}' \end{aligned}$$

where Φ_0 and $\vec{A}_0$ satisfy $\Box\Phi_0 = 0$ and $\Box\vec{A}_0 = \vec{0}$.

We showed in tutorial that one such Green's function is

$$G(t, \vec{r}; t', \vec{r}') \quad = \quad -\frac{1}{4\pi}\,\frac{\delta\left(t - t' - \frac{|\vec{r} - \vec{r}'|}{c}\right)}{|\vec{r} - \vec{r}'|}$$

so this gives the scalar potential as

$$\begin{aligned} \Phi(t, \vec{r}) &= \Phi_0(t, \vec{r}) + \frac{1}{4\pi\epsilon_0}\int \frac{\rho(t', \vec{r}')}{|\vec{r} - \vec{r}'|}\,\delta\left(t - t' - \frac{|\vec{r} - \vec{r}'|}{c}\right)\, \mathrm{d}t'\, \mathrm{d}^3\vec{r}' \\ &= \Phi_0(t, \vec{r}) + \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(t - \frac{|\vec{r} - \vec{r}'|}{c}, \vec{r}'\right)}{|\vec{r} - \vec{r}'|}\, \mathrm{d}^3\vec{r}' \end{aligned}$$

because the argument of the delta-function is zero when $t' = t - |\vec{r} - \vec{r}'|/c$.

But why this Green's function? They aren't unique, so why this one and not another? For example, the function

$$G_+(t, \vec{r}; t', \vec{r}') \;\; = \;\; -\frac{1}{4\pi} \frac{\delta\left(t - t' + \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r} - \vec{r}'|}$$

also satisfies Green's equation, and if we use this instead and follow the same steps as above, we'd get a scalar potential

$$\Phi_+(t, \vec{r}) \;\; = \;\; \Phi_0(t, \vec{r}) + \frac{1}{4\pi\epsilon_0} \int \frac{\rho\left(t + \frac{|\vec{r}-\vec{r}'|}{c}, \vec{r}'\right)}{|\vec{r} - \vec{r}'|} \, \mathrm{d}^3\vec{r}'.$$

Mathematically, these both solve $\Box\Phi = -\rho/\epsilon_0$. But it's *physics* that tells us the first is correct and the second is wrong.

Look at the time-dependence in the first expression for Φ; $t - |\vec{r} - \vec{r}'|/c$ appears in ρ. What this indicates is that at the time t we're measuring Φ, the contribution due to the little bit of charge at $\vec{r}'$ is from the *past*, specifically a time $|\vec{r} - \vec{r}'|/c$ before t. This is fine; we fully expect the present potential to be due to what the charges were doing a little while ago, since it takes a bit of time for the information to reach us. In fact, it's exactly what we expect: $|\vec{r} - \vec{r}'|$ is the distance between us and the charge, and if the info travels at the speed of light, $|\vec{r} - \vec{r}'|/c$ is precisely the time it would take.

In contrast, in Φ_+, it's $t + |\vec{r} - \vec{r}'|/c$ that's in the density. But this is in the *future*, meaning the potential in the present is determined by things the charges *haven't done yet*. In short, we have a violation of causality in Φ_+, and since cause-precedes-effect seems to be built into the universe, we discard this solution on the basis of physics.

In fact, the Green's function

$$G(t, \vec{r}; t', \vec{r}') \;\; = \;\; -\frac{1}{4\pi} \frac{\delta\left(t - t' - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r} - \vec{r}'|}$$

is the only one that respects causality, and so this is the one we must use whenever solving a physics problem involving the d'Alembertian. The time $t - |\vec{r} - \vec{r}'|/c$ is sometimes called the *retarded time* t_{ret} and this particular function the *retarded Green's function*. The other solution G_+ is similarly referred to as the *advanced Green's function* and $t_{\rm ad} = t + |\vec{r} - \vec{r}'|/c$ as the *advanced time.*

And notice something revolutionary has just happened. This is supposed to be *classical* physics, in the sense of what Maxwell and his colleagues knew in the mid-to-late 19$^{\rm th}$ Century. But somehow concepts – like causality – related to *relativity* have shown up. This will be elaborated upon in the final part of this module a few lectures from now...

So we've found a solution for the scalar field, and since we now know which Green's function to use, we use the same sorts of calculations to get the vector potential as well. Thus, the most general physical solutions to the potential equations are

$$\begin{aligned} \Phi(t,\vec{r}) &= \Phi_0(t,\vec{r}) + \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(t - \frac{|\vec{r}-\vec{r}'|}{c}, \vec{r}'\right)}{|\vec{r}-\vec{r}'|}\, {\rm d}^3\vec{r}', \\ \vec{A}(t,\vec{r}) &= \vec{A}_0(t,\vec{r}) + \frac{\mu_0}{4\pi}\int \frac{\vec{J}\left(t - \frac{|\vec{r}-\vec{r}'|}{c}, \vec{r}'\right)}{|\vec{r}-\vec{r}'|}\, {\rm d}^3\vec{r}'. \end{aligned}$$

Now, Φ_0 and $\vec{A}_0$ satisfy the source-free equations and are determined by whatever boundary conditions we may have. However, if we specifically consider cases of localised charges and currents, then $\Phi \to 0$ and $\vec{A} \to \vec{0}$ as $|\vec{r}| \to \infty$ can be imposed and this leads to Φ_0 and $\vec{A}_0$ both vanishing. Thus, in the cases we'll be looking at now, we'll use

$$\begin{aligned} \Phi(t,\vec{r}) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(t - \frac{|\vec{r}-\vec{r}'|}{c}, \vec{r}'\right)}{|\vec{r}-\vec{r}'|}\, {\rm d}^3\vec{r}', \\ \vec{A}(t,\vec{r}) &= \frac{\mu_0}{4\pi}\int \frac{\vec{J}\left(t - \frac{|\vec{r}-\vec{r}'|}{c}, \vec{r}'\right)}{|\vec{r}-\vec{r}'|}\, {\rm d}^3\vec{r}' \end{aligned}$$

as the formulae giving our potentials.

(For consistency's sake, we should now check that the above satisfy the Lorentz gauge condition, but we won't do that here. I may do it in next week's tutorial, but if I don't manage to do that, try it yourself. You'll see that the gauge condition is indeed satisfied.)

B. EM Fields for Localised Oscillatory Sources

So we have expressions for the scalar and vector potentials for the general time-dependent case, and these of course give the electric and magnetic fields. At least, they would if we could actually calculate them, but only for very simple, highly symmetric systems do analytic solutions exist, so we now need to ask what we can do in more complicated cases.

The answer is, of course, we approximate. There will be lots of similarity between what we do now and what we did when doing the multipole expansions in the static case. However, we'll find that the time-dependence makes things *much* more interesting, and we'll actually be able to derive the basic principles behind *telecommunication*, which is, after all, the study of how to manipulate EM fields to transmit information over long distances.

Our first approximation is that the time-dependence of all of our sources is simply oscillatory; more specifically, our current density has the form

$$\vec{J}(t,\vec{r}) = \mathrm{Re}\left[\tilde{\vec{J}}(\vec{r})e^{-i\omega t}\right]$$

for some frequency ω and complex function $\tilde{\vec{J}}(\vec{r})$. This immediately gives the charge density via the continuity equation:

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= -\vec{\nabla}\cdot\vec{J} \\ &= -\mathrm{Re}\left[\vec{\nabla}\cdot\tilde{\vec{J}}e^{-i\omega t}\right]\end{aligned}$$

and if we integrate this with respect to t, we see that

$$\begin{aligned}\rho(t,\vec{r}) &= \mathrm{Re}\left[\tilde{\rho}(\vec{r})e^{-i\omega t}\right], \\ \tilde{\rho}(\vec{r}) &= -\frac{i}{\omega}\vec{\nabla}\cdot\tilde{\vec{J}}(\vec{r})\end{aligned}$$

so we only need to specify $\vec{J}$ to get ρ.

Now we put these two sources into our expressions for the potentials; for the vector potential, we find

$$\begin{aligned}\vec{A}(t,\vec{r}) &= \frac{\mu_0}{4\pi}\int\frac{\mathrm{Re}\left[\tilde{\vec{J}}(\vec{r}')e^{-i\omega(t-|\vec{r}-\vec{r}'|/c)}\right]}{|\vec{r}-\vec{r}'|}\,\mathrm{d}^3\vec{r}' \\ &= \mathrm{Re}\left[\tilde{\vec{A}}(\vec{r})e^{-i\omega t}\right]\end{aligned}$$

where

$$\tilde{\vec{A}}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\tilde{\vec{J}}(\vec{r}')e^{i\omega|\vec{r}-\vec{r}'|/c}}{|\vec{r}-\vec{r}'|}\,\mathrm{d}^3\vec{r}'.$$

The scalar potential can be found from either the expression in terms of ρ or the Lorentz gauge condition $\partial\Phi/\partial t = -c^2\vec{\nabla}\cdot\vec{A}$; either way, we find $\Phi(t,\vec{r}) = \mathrm{Re}[\tilde{\Phi}(\vec{r})e^{-i\omega t}]$ where

$$\tilde{\Phi} = -\frac{ic^2}{\omega}\vec{\nabla}\cdot\tilde{\vec{A}}.$$

(Again, everything is given once $\tilde{\vec{J}}$ is picked.)

Now, we're not talking about propagating waves (yet), but just for notational convenience, we'll still define the wave number k to $k = \omega/c$. This means that one of the factors in the integrand giving $\tilde{\vec{A}}$ is $e^{ik|\vec{r}-\vec{r}'|}/|\vec{r}-\vec{r}'|$, and we want to Taylor-expand this as a function of the primed variables around $\vec{r}' = \vec{0}$. The motivation for doing this is the same as in the static case: if the charge/current configuration is localised, it'll have some characteristic size L so that all the integration variables have $|\vec{r}'| = r' \leq L$. Thus, if we look at positions with $|\vec{r}| = r \gg L$, r'/r will be a very small parameter and expanding in terms of this parameter will give us a way to approximate our potentials and fields. However, we also have *another* length scale in the system, namely the effective wavelength $\lambda = 2\pi/k$. This is what distinguishes the time-dependent situation from the static one (where we only had a single length scale). We'll see shortly that we'll need to consider how r and λ compare in size when we try to approximate our potentials and fields.

To make the notation a bit easier, define $f(\vec{r}) = e^{ik|\vec{r}|}/|\vec{r}|$; thus, we want the Taylor series expansion of $f(\vec{r}-\vec{r}')$ around $\vec{r}' = \vec{0}$. The first three terms in this expansion are

$$\begin{aligned} f(\vec{r}-\vec{r}') &= [f(\vec{r}-\vec{r}')|_{\vec{r}'=\vec{0}} + \sum_i [\partial_i' f(\vec{r}-\vec{r}')|_{\vec{r}'=\vec{0}}\, x_i' \\ &\quad + \frac{1}{2}\sum_{i,j}\left[\partial_i'\partial_j' f(\vec{r}-\vec{r}')\right|_{\vec{r}'=\vec{0}}\, x_i' x_j' + \ldots \end{aligned}$$

where $\partial_i' = \partial/\partial x_i'$. This looks a bit complicated, but we can actually simplify it quite nicely with this trick: we've noted before that if we have a function that depends on the difference between two variables, the derivative with respect to one is the negative of the derivative with respect to the other. In this case, $\partial'_i f(\vec{r}-\vec{r}\,') = -\partial_i f(\vec{r}-\vec{r}\,')$. Taking another derivative picks up another minus sign, so $\partial'_i\partial'_j f(\vec{r}-\vec{r}\,') = \partial_i\partial_j f(\vec{r}-\vec{r}\,')$, and so on. These are exactly the types of derivatives which appear in the above expansion. Now, after taking the derivative, we set $\vec{r}\,'$ to zero, but using the above identities, we see that because we're replaced all $\vec{r}\,'$-derivatives by ones with respect to $\vec{r}$, we may set $\vec{r}\,'$ to zero *before* taking the $\vec{r}$-derivatives, so that $\partial'_i f(\vec{r}-\vec{r}\,')$ at $\vec{r}\,' = \vec{0}$ is just $-\partial_i f(\vec{r})$ and so on. More generally, the n^{th}-derivative appearing in the expansion cen be written as

$$\left[\partial'_{i_1}\partial'_{i_2}\ldots\partial'_{i_n} f(\vec{r}-\vec{r}\,')\right|_{\vec{r}\,'=\vec{0}} \;=\; (-1)^n \partial_{i_1}\partial_{i_2}\ldots\partial_{i_n} f(\vec{r}).$$

This is true for any function f, but the specific function we consider here is particularly easy when we express it in spherical coordinates: $f(\vec{r}) = e^{ikr}/r$. Using all of this therefore tells us that

$$\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \;=\; \frac{e^{ikr}}{r} - \sum_i \frac{\partial}{\partial x_i}\left(\frac{e^{ikr}}{r}\right) x'_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{e^{ikr}}{r}\right) x'_i x'_j + \ldots$$

We know that $\partial_i r = x_i/r$, and, using this and some straightforward calculus, we find that

$$\begin{aligned}
\frac{\partial}{\partial x_i}\left(\frac{e^{ikr}}{r}\right) &= (ikr-1)x_i\frac{e^{ikr}}{r^3}, \\
\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{e^{ikr}}{r}\right) &= [3(1-ikr)x_i x_j - k^2 r^2 x_i x_j - r^2(1-ikr)\delta_{ij}]\frac{e^{ikr}}{r^5}
\end{aligned}$$

and higher derivatives would be computed similarly.

Now, with all the above tricks, we could in principle compute all terms in the expansion by taking derivatives of e^{ikr}/r. However, here we'll only consider the leading-order term, as it's the one that dominates when r is large compared to the size of the charge distribution. Therefore, the vector potential is

$$\begin{aligned}
\vec{A}(t,\vec{r}) &= \frac{\mu_0}{4\pi}\int \frac{\mathrm{Re}\left[\tilde{\vec{J}}(\vec{r}\,')e^{-i\omega(t-|\vec{r}-\vec{r}\,'|/c)}\right]}{|\vec{r}-\vec{r}\,'|}\,\mathrm{d}^3\vec{r}\,' \\
&= \mathrm{Re}\left[\frac{\mu_0}{4\pi}\int \tilde{\vec{J}}(\vec{r}\,')e^{-i\omega t}\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\,\mathrm{d}^3\vec{r}\,'\right] \\
&= \mathrm{Re}\left\{\frac{\mu_0}{4\pi}\frac{e^{i(kr-\omega t)}}{r}\int \tilde{\vec{J}}(\vec{r}\,')\left[1+O\left(\frac{r'}{r}\right)\right]\mathrm{d}^3\vec{r}\,'\right\} \\
&\approx \mathrm{Re}\left[\frac{\mu_0}{4\pi}\frac{e^{i(kr-\omega t)}}{r}\int \tilde{\vec{J}}(\vec{r}\,')\,\mathrm{d}^3\vec{r}\,'\right].
\end{aligned}$$

Notice the prefactor $e^{i(kr-\omega t)}/r$; note that if $\vec{k}=k\hat{e}_r$, then $kr-\omega t=\vec{k}\cdot\vec{r}-\omega t$, so this describes a wave propagating radially away from the origin, with an amplitude that falls off as $1/r$. Such a wave is called a *spherical wave* because it solves the wave equation for a spherically-symmetric system. Thus, a localised oscillating charge configuration sends out a spherical EM wave.

Let's now compute the vector potential above; its i^{th}-component is

$$A_i(t,\vec{r}) \approx \mathrm{Re}\left[\frac{\mu_0}{4\pi}\frac{e^{i(kr-\omega t)}}{r}\int \tilde{J}_i(\vec{r}\,')\,\mathrm{d}^3\vec{r}\,'\right]$$

and we've seen an integral like this before. Recall that quite a while ago we proved the identity

$$\tilde{J}_i(\vec{r}\,') = \vec{\nabla}'\cdot\left[x_i'\tilde{\vec{J}}(\vec{r}\,')\right]-x_i'\vec{\nabla}'\cdot\tilde{\vec{J}}(\vec{r}\,').$$

When we derived this, we were looking at the static case, where the divergence of $\vec{J}$ vanished, and so the last term in the above identity was zero. That's *not* the case now, however; we showed that $\vec{\nabla}\cdot\tilde{\vec{J}}=i\omega\tilde{\rho}$, so

$$\begin{aligned}
\int \tilde{J}_i(\vec{r}\,')\,\mathrm{d}^3\vec{r}\,' &= \int\left\{\vec{\nabla}'\cdot\left[x_i'\tilde{\vec{J}}(\vec{r}\,')\right]-i\omega x_i'\tilde{\rho}(\vec{r}\,')\right\}\mathrm{d}^3\vec{r}\,' \\
&= -i\omega\int x_i'\tilde{\rho}(\vec{r}\,')\,\mathrm{d}^3\vec{r}\,'.
\end{aligned}$$

(As usual, we've assumed that all sources vanish at infinity, so the surface integral we get from the first term above vanishes.) This is proportional to the *complex electric dipole amplitude*, and here's why: the i^{th}-component of the electric dipole moment is

$$\begin{aligned} p_i(t) &= \int x_i' \rho(t, \vec{r}')\, \mathrm{d}^3\vec{r}' \\ &= \mathrm{Re}\left[e^{-i\omega t} \int x_i' \tilde{\rho}(\vec{r}')\, \mathrm{d}^3\vec{r}'\right] \\ &= \mathrm{Re}\left[e^{-i\omega t} (\tilde{\vec{p}}_0)_i\right] \end{aligned}$$

where

$$\tilde{\vec{p}}_0 = \int \vec{r}' \tilde{\rho}(\vec{r}')\, \mathrm{d}^3\vec{r}'.$$

(We use a zero subscript to drive home the point that this is a constant vector.) Therefore,

$$\int \tilde{\vec{J}}(\vec{r}')\, \mathrm{d}^3\vec{r}' = -i\omega \tilde{\vec{p}}_0.$$

(Alternatively, since we assume $\tilde{\vec{J}}$ is given, this says the complex electric dipole is just the integral of $i\tilde{\vec{J}}/\omega$.) Thus, our vector potential is

$$\vec{A}(t, \vec{r}) \approx \frac{\mu_0}{4\pi} \mathrm{Re}\left[-\frac{i\omega \tilde{\vec{p}}_0\, e^{i(kr-\omega t)}}{r}\right].$$

From this, we can use the Lorentz gauge condition to find the scalar potential, or at least its complex amplitude:

$$\begin{aligned} \tilde{\Phi}(\vec{r}) &= -\frac{ic^2}{\omega} \vec{\nabla} \cdot \tilde{\vec{A}}(\vec{r}) \\ &\approx -\frac{ic^2}{\omega} \vec{\nabla} \cdot \left(\frac{-i\omega\mu_0}{4\pi} \frac{e^{ikr}}{r} \tilde{\vec{p}}_0\right) \\ &= \frac{1}{4\pi\epsilon_0} \frac{(1 - ikr)(\vec{r} \cdot \tilde{\vec{p}}_0) e^{ikr}}{r^3} \end{aligned}$$

where we've again used $c^2 = 1/\mu_0\epsilon_0$. Thus,

$$\begin{aligned} \Phi(t, \vec{r}) &= \mathrm{Re}\left[\tilde{\Phi}(\vec{r}) e^{-i\omega t}\right] \\ &\approx \frac{1}{4\pi\epsilon_0} \mathrm{Re}\left[\frac{(1 - ikr)(\vec{r} \cdot \tilde{\vec{p}}_0) e^{i(kr-\omega t)}}{r^3}\right]. \end{aligned}$$

But, of course, it's the electric and magnetic fields we actually want. We see that all the sources oscillate with the same frequency ω, so the fields will as well and thus we can write

$$\vec{E}(t,\vec{r}) = \mathrm{Re}\left[\tilde{\vec{E}}(\vec{r})e^{-i\omega t}\right], \qquad \vec{B}(t,\vec{r}) = \mathrm{Re}\left[\tilde{\vec{B}}(\vec{r})e^{-i\omega t}\right].$$

Now, we actually only need the magnetic field; since we're assuming that we're far away from any sources, $\vec{J} = \vec{0}$ in this region and thus Ampère's law gives

$$\begin{aligned}
\vec{\nabla} \times \vec{B} &= \mathrm{Re}\left[(\vec{\nabla} \times \tilde{\vec{B}})e^{-i\omega t}\right] \\
&= \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \\
&= \mathrm{Re}\left[-\frac{i\omega}{c^2}\tilde{\vec{E}}e^{-i\omega t}\right]
\end{aligned}$$

and therefore we get the electric field amplitude in terms of the magnetic field:

$$\tilde{\vec{E}} = \frac{ic^2}{\omega}\vec{\nabla} \times \tilde{\vec{B}}.$$

Now for the magnetic field: $\vec{B} = \vec{\nabla} \times \vec{A}$ implies $\tilde{\vec{B}} = \vec{\nabla} \times \tilde{\vec{A}}$, or

$$\begin{aligned}
\tilde{\vec{B}} &\approx \vec{\nabla} \times \left(\frac{-i\omega\mu_0}{4\pi}\frac{e^{ikr}}{r}\tilde{\vec{p}}_0\right) \\
&= \frac{i\omega\mu_0\tilde{\vec{p}}_0}{4\pi} \times \vec{\nabla}\left(\frac{e^{ikr}}{r}\right) \\
&= \frac{\mu_0}{4\pi}(ikr-1)\left(\frac{i\omega\tilde{\vec{p}}_0 \times \vec{r}\, e^{ikr}}{r^3}\right).
\end{aligned}$$

This is an approximation that's good for all values of r that are much greater than the size of our charge distribution. However, now we need to consider the other length scale mentioned earlier, the wavelength $\lambda = 2\pi/k = 2\pi c/\omega$. We see that $kr = 2\pi r/\lambda$, so how we treat this term depends on how r and λ compare to one another, and this breaks our analysis into two regimes:

1. The Near-Zone Approximation

We consider first the case where kr is very small, i.e. $r \ll \lambda$; even though r is large compared to the size of the distribution, it's small compared to the wavelength of the oscillation. This is called the "near zone", and in this region, we see that we can ignore the kr term and so the complex amplitude of the magnetic field is approximately

$$\tilde{\vec{B}} \approx -\frac{\mu_0}{4\pi}\left(\frac{i\omega\tilde{\vec{p}}_0 \times \vec{r}\, e^{ikr}}{r^3}\right)$$

and so the electric field amplitude is

$$\begin{aligned}\tilde{\vec{E}} &\approx \frac{ic^2}{\omega}\vec{\nabla}\times\left[-\frac{\mu_0}{4\pi}\left(\frac{i\omega\tilde{\vec{p}}_0 \times \vec{r}\, e^{ikr}}{r^3}\right)\right] \\ &= \frac{1}{4\pi\epsilon_0}\vec{\nabla}\times\left(\frac{\tilde{\vec{p}}_0 \times \vec{r}\, e^{ikr}}{r^3}\right).\end{aligned}$$

Now, any time the derivative acts on e^{ikr}, it'll pull down a factor of k. However, since the wavelength is so large, terms with this factor will be much smaller than terms without it and so we may treat the exponential as a constant in the near zone, and thus

$$\tilde{\vec{E}} \approx \frac{e^{ikr}}{4\pi\epsilon_0}\vec{\nabla}\times\left(\frac{\tilde{\vec{p}}_0 \times \vec{r}}{r^3}\right).$$

But we've already done this calculation! Recall that in magnetostatics, the dipole contribution to the vector potential was proportional to $\vec{m}\times\vec{r}/r^3$, and the magnetic field is just the curl of this, which is exactly the calculation we have here, and the result is the same: a dipole field of the form

$$\tilde{\vec{E}} \approx \frac{e^{ikr}}{4\pi\epsilon_0}\frac{3(\tilde{\vec{p}}_0\cdot\vec{r})\vec{r} - r^2\tilde{\vec{p}}_0}{r^5}.$$

Now, an important point, especially in contrast to what we'll see in the next lecture: Note that the magnetic field goes as $1/r^2$ as r gets big, and we see that the electric field goes as $1/r^3$, and so the Poynting vector – which gives the power flux – goes as $1/r^5$. Now, suppose we surround the source with a very large sphere, pick a small area on this sphere and ask, what's the power flowing through this area? The surface area element on a sphere of radius r is $\mathrm{d}\vec{\sigma} = r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \hat{e}_r$, so the bit of power flowing through it is $\vec{S} \cdot \mathrm{d}\vec{\sigma}$, which goes as $1/r^5$ times r^2, or $1/r^3$. Thus, as the sphere gets larger, the power flowing through it decreases very rapidly, and thus drops to zero in the limit of an infinitely-large sphere.

Maybe this doesn't seem too surprising – we expect fields to fall off as r increases – but as we'll see in the next topic, something *very* different happens to the radiated power in the case where kr is large, and this difference is at the very core of modern telecommunications. But that's for next week...

Side comment: although this assumption seems restrictive, it's actually not. Keep in mind that any time-dependent function $f(t)$ may be written as

$$f(t) \;=\; \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t} \mathrm{d}t$$

where (up to some numerical factor) $\tilde{f}(\omega)$ is the Fourier transform of $f(t)$. So if we interpret $\tilde{\vec{J}}$ as the Fourier transform of $\vec{J}(t, \vec{r})$ with respect to t, we can do all calculations as if the time-dependence was purely $e^{-i\omega t}$. Then by inverse Fourier-transforming all results with respect to ω, we'll obtain expressions valid for any time-dependence of $\vec{J}$. However, we'll for the most part assume a simple oscillatory behaviour to simplify the calculations.