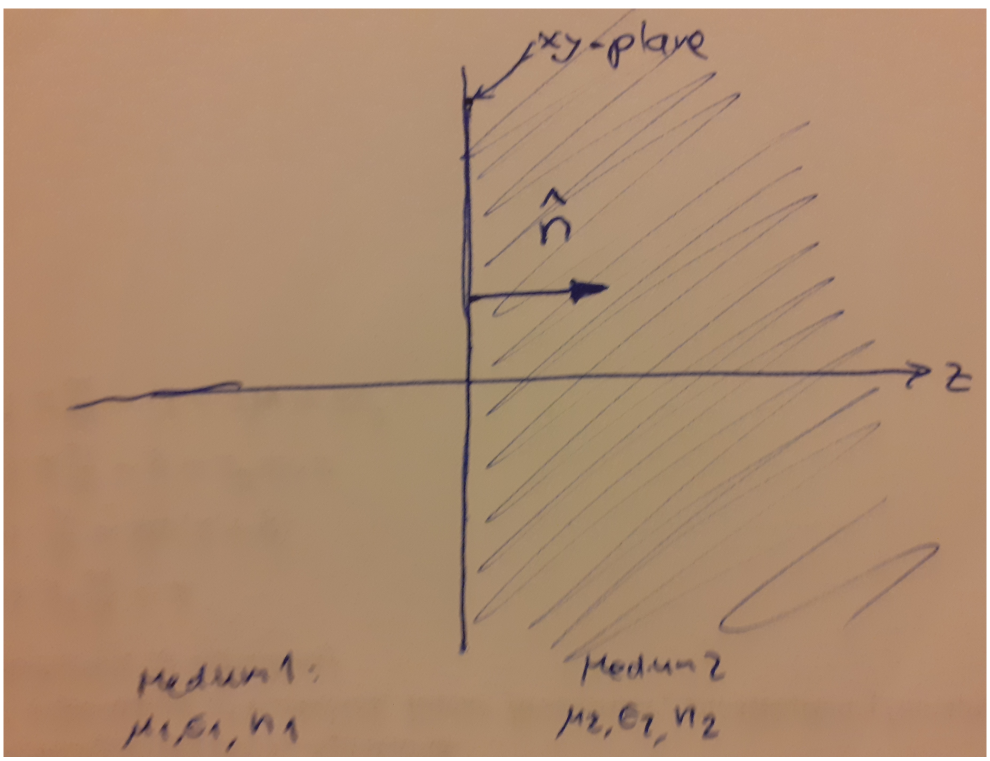
**MP465 – Advanced Electromagnetism**

**Lectures 17 & 18 Part I (9 April 2020)**

# C. Plane Waves in across Boundaries: Reflection and Refraction

## 1. Boundary Conditions

Everything we've talked about in this section up to this point has all been within a *single* linear medium, but now we look at what happens when we have two linear media. Within either one of them, we know most everything about plane wave solutions, but what about if a plane wave goes from one medium into another? What happens then?




To answer these questions, we have to look at what goes on at the boundary between the two media. To do so, we need to list what we know: first off, each media has its own permittivity, permeability and index of refraction, $\epsilon_1$, $\mu_1$ and $n_1 = \sqrt{\mu_1\epsilon_1/\mu_0\epsilon_0}$ for medium 1, and similarly for medium 2.

Next, we assume the boundary is smooth, so at any point on the boundary it looks flat if we zoom up close. Thus, we can always locally pick a Cartesian coordinate system such the the boundary is in the $xy$-plane, with medium 1 in the $z < 0$ region and medium 2 in the $z > 0$ region, as shown on the previous page.

If we continue to assume there are no free charges in the system, then we have the same Maxwell equations as before, except this time we'll write them in integral form: for any closed surface $\Sigma$,

$$\oint_\Sigma \vec{D} \cdot \mathrm{d}\vec{\sigma} = 0, \qquad \oint_\Sigma \vec{B} \cdot \mathrm{d}\vec{\sigma} = 0,$$

and for any open surface $\mathcal{S}$ with boundary (a closed curve) $\mathcal{C}$,

$$\oint_\mathcal{C} \vec{E} \cdot \mathrm{d}\vec{r} = -\frac{\partial}{\partial t}\int_\mathcal{S} \vec{B} \cdot \mathrm{d}\vec{\sigma}, \qquad \oint_\mathcal{C} \vec{H} \cdot \mathrm{d}\vec{r} = \frac{\partial}{\partial t}\int_\mathcal{S} \vec{D} \cdot \mathrm{d}\vec{\sigma},$$

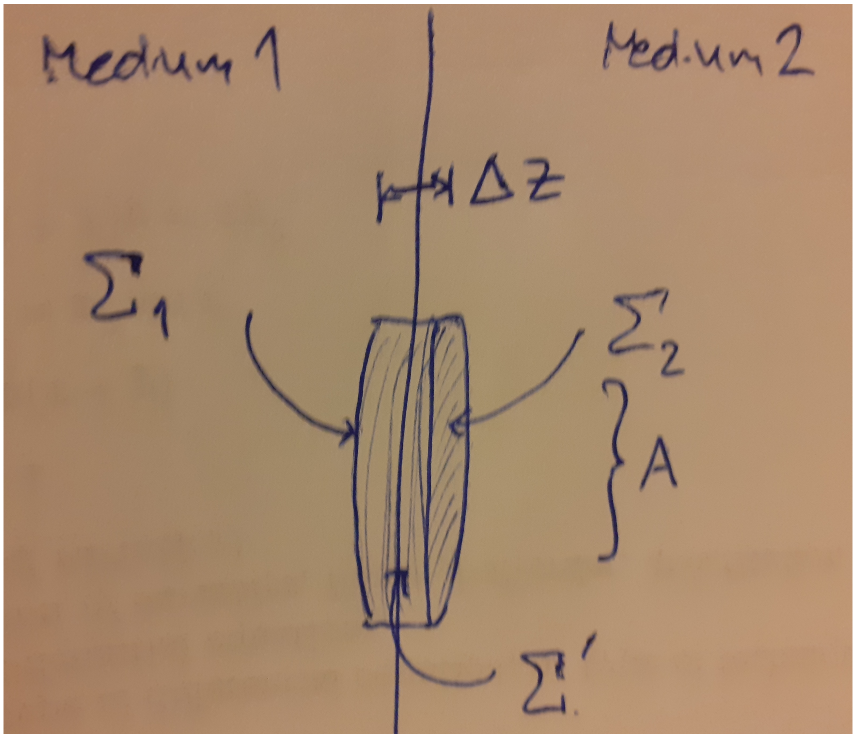
Now, let's look at what the first of these implies: suppose we take $\Sigma$ to be a little cylinder straddling the boundary, as shown on the next page. The two caps are $\Sigma_1$ in medium 1 and $\Sigma_2$ in medium 2, and the cylindrical shell $\Sigma'$ crosses the boundary. Obviously,

$$\oint_\Sigma \vec{D} \cdot \mathrm{d}\vec{\sigma} \;=\; \int_{\Sigma_1} \vec{D} \cdot \mathrm{d}\vec{\sigma} + \int_{\Sigma_2} \vec{D} \cdot \mathrm{d}\vec{\sigma} + \int_{\Sigma'} \vec{D} \cdot \mathrm{d}\vec{\sigma}$$

but because the length of $\Sigma'$, $\Delta z$, is assumed to be extremely small, the last of these integrals will be negligible, leaving only the first two. Now, on $\Sigma_1$, the unit normal is $-\hat{e}_z$ and if the cross-sectional area $A$ is small, we expect the displacement field to be approximately constant on $\Sigma_1$; let's call it $\vec{D}_1$. All of this gives the first of the above integrals to be $-D_{1z}A$. On $\Sigma_2$, the unit normal is $\hat{e}_z$, so an analagous argument gives $D_{2z}A$ for the second integral. Thus, we have the condition $-D_{1z}A + D_{2z}A = 0$, or $D_{1z} = D_{2z}$. More generally, this may be written as

$$\left(\vec{D}_2 - \vec{D}_1\right)_\perp\Big|_{\text{boundary}} \;=\; 0.$$

In other words, *the component of the electric displacement field perpendicular to the boundary must be the same on either side of the boundary.*




Now, the second of Maxwell's equations above has exactly the same form except with $\vec{D}$ replaced by $\vec{B}$, so the same argument can be applied with an analagous result:

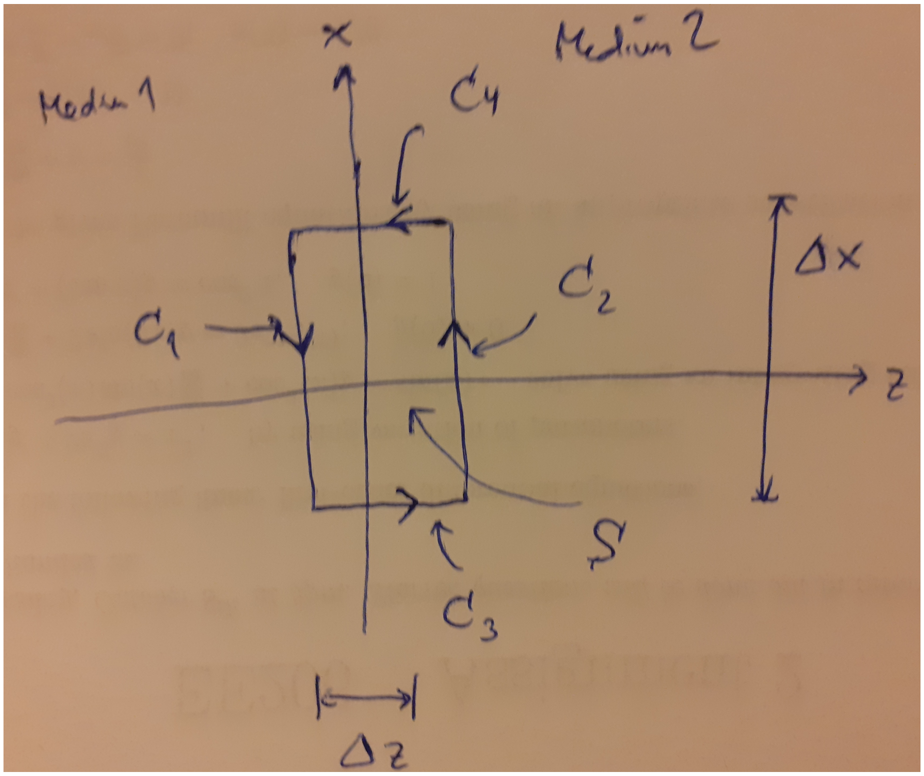
$$\left(\vec{B}_2 - \vec{B}_1\right)_\perp\Big|_{\text{boundary}} = 0$$

or *the component of the magnetic field perpendicular to the boundary must be the same on either side of the boundary.*

Let's now go to the third Maxwell equation above and see what it says: take the surface $\mathcal{S}$ to be a little rectangular region depicted below parallel to the $xz$-plane straddling the boundary with width $\Delta z$ and height $\Delta x$, both assumed to be very small.

Then we expect the surface integral $\int_\mathcal{S} \vec{B} \cdot \mathrm{d}\vec{\sigma}$ to be of size $\Delta x \Delta z$, which will go to zero as we shrink $\Delta z$ to zero. Thus, for this particular case, we expect $\oint_\mathcal{C} \vec{E} \cdot \mathrm{d}\vec{r}$ to be very close to zero.

But this line integral can be broken up into four integrals each over the individual sides. The integrals over $\mathcal{C}_3$ and $\mathcal{C}_4$ will each be of order $\Delta z$ and




thus will become negligible in the $\Delta z \to 0$ limit. On $\mathcal{C}_1$, we see $\mathrm{d}\vec{r} = -\hat{e_x}\mathrm{d}x$, so if we say that the electric field just to the left of the boundary is $\vec{E}_1$, then this integral gives $-E_{1x}\Delta x$ as its value. Similarly, the integral over $\mathcal{C}_2$ will give $E_{2x}\Delta x$. Thus, we get the result that $-E_{1x}\Delta x + E_{2x}\Delta x = 0$, or $E_{1x} = E_{2x}$.

But we could have just as well taken $\mathcal{S}$ to have been a rectangle straddling the boundary parallel to the $yz$-plane; if this rectangle has side lengths $\Delta y$ and $\Delta Z$, then the exact same type of argument as above will give $E_{1y} = E_{2y}$. But the $x$- and $y$-components are precisely those *parallel* to the boundary, which gives us the generalised result

$$\left(\vec{E}_2 - \vec{E}_1\right)_{\parallel}\Big|_{\text{boundary}} = 0.$$

In other words, *the components of the electric field parallel to the boundary must be the same on either side of the boundary.*

Finally, the last Maxwell equation, but we're already done the work: if we repeat the preceding argument with $\vec{E}$ replaced by $\vec{H}$ and $\vec{B}$ replaced by

$-\vec{D}$, we get

$$\left(\vec{H}_2 - \vec{H}_1\right)_{\parallel}\Big|_{\text{boundary}} = 0.$$

or *the components of the magnetic intensity field parallel to the boundary must be the same on either side of the boundary.*

Let's restate all of this again, because these results lie at the heart of any discussion of EM waves travelling between two different media. If we assume that there are no free charges or currents, then the following have to be continuous at the boundary, i.e. have the same value regardless of which side of the boundary we're on:

- $D_\perp$, the component of the electric displacement field normal to the boundary;
- $B_\perp$, the component of the magnetic field normal to the boundary;
- $\vec{E}_\parallel$, the two components of the electric field parallel to the boundary;
- $\vec{H}_\parallel$, the two components of the magnetic intensity field parallel normal to the boundary;

(In the case where the boundary lies in the $xy$-plane, these are equivalent to the continuity of $D_z$, $B_z$, $E_x$, $E_y$, $H_x$ and $H_y$ at $z = 0$.)