MP465 – Advanced Electromagnetism

Lectures 13 & 14 (25 March 2020)

A. Maxwell's Equations in Matter (continued)

In the last lecture, we introduced the polarisability (also called the electric polarisation, but we'll save "polarisation" for another unrelated concept that'll be showing up in a few lectures) $\vec{P}$ of a medium, which is basically an electric dipole density due to the charges bound to the constituents making up the medium. We argued that the contribution of this to the scalar potential should be

$$\Phi_P(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\vec{P}(\vec{r}\,')\cdot(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,\mathrm{d}^3\vec{r}\,'$$

By using the identity

$$\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = \vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}\,'|}$$

we can use the "reverse product rule" and Gauss' theorem to obtain

$$\Phi_P(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint_\Sigma \frac{\vec{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\cdot \mathrm{d}\vec{\sigma}\,' - \frac{1}{4\pi\epsilon_0}\int \frac{\vec{\nabla}'\cdot\vec{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,\mathrm{d}^3\vec{r}\,'$$

where Σ is the surface of the medium under consideration. The contribution $\vec{E}_P$ of the polarisability to the electric field is the negative gradient of this, but we're interested in the divergence of $\vec{E}_P$. Why? Well, we know that the total electric field $\vec{E}$ is related to the total charge density ρ_{tot} by $\vec{\nabla}\cdot\vec{E} = \rho_{tot}/\epsilon_0$. so it follows that if $\vec{E}_P$ is entirely due to the bound charges, then $\vec{\nabla}\cdot\vec{E}_P = \rho_b/\epsilon_0$, where ρ_b is the bound charge density.

So if we want the divergence of this field, we use $\vec{\nabla}\cdot\vec{E}_P = -\nabla^2\Phi_P$ to obtain

$$\begin{aligned}\vec{\nabla}\cdot\vec{E}_P(\vec{r}) &= -\nabla^2\Phi_P(\vec{r}) \\ &= -\frac{1}{4\pi\epsilon_0}\oint_\Sigma\left(\nabla^2\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\vec{P}(\vec{r}\,')\cdot\mathrm{d}\vec{\sigma}\,' + \frac{1}{4\pi\epsilon_0}\int\left(\nabla^2\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\vec{\nabla}'\cdot\vec{P}(\vec{r}\,')\,\mathrm{d}^3\vec{r}\,'\end{aligned}$$

since the laplacian is with respect to the unprimed variables. We've seen the quantity in the brackets loads of times before: it's $-4\pi\delta^{(3)}(\vec{r}-\vec{r}')$, so

$$\vec{\nabla}\cdot\vec{E}_P(\vec{r}) = \frac{1}{\epsilon_0}\oint_\Sigma \delta^{(3)}(\vec{r}-\vec{r}')\vec{P}(\vec{r}')\cdot \mathrm{d}\vec{\sigma}' - \frac{1}{\epsilon_0}\int \delta^{(3)}(\vec{r}-\vec{r}')\vec{\nabla}'\cdot\vec{P}(\vec{r}')\,\mathrm{d}^3\vec{r}'.$$

Now, we know what to do if we have a *volume* integral with a delta-function in it, but what do we do if it's a *surface* integral? The above expression involves both.

In general, it depends on the surface (not surprisingly). However, if we decide that we're not interested in what's happening right on the surface of the medium, but only in its exterior or interior, then the surface integral vanishes. Why? Well, all the points of integration (i.e. all values of the position vector $\vec{r}'$) lie on Σ, but if the position where we're looking – $\vec{r}$ – isn't on Σ, then the argument of the delta-function in the surface integral is never zero and so the delta-function vanishes, leaving us only with the volume integral.

And that we *can* do, and we get the result

$$\vec{\nabla}\cdot\vec{E}_P(\vec{r}) = \rho_b(\vec{r})/\epsilon_0 = -\frac{1}{\epsilon_0}\vec{\nabla}\cdot\vec{P}(\vec{r}),$$

or

$$\vec{\nabla}\cdot\vec{P} = -\rho_b.$$

So if we know the bound charge density, then we can say something about the polarisability. But we *don't* in general know ρ_b except for very simple media (like, say, a monatomic ideal gas). It's the *free* charge density ρ that we have control over or some way of measuring. Does that give us anything?

Yep; $\rho_{tot} = \rho_b + \rho$ (a charge can only be free or bound), so

$$\begin{aligned}\rho = \rho_{tot} - \rho_b &= \epsilon_0\vec{\nabla}\cdot\vec{E} + \vec{\nabla}\cdot\vec{P} \\ &= \vec{\nabla}\cdot\left(\epsilon_0\vec{E} + \vec{P}\right)\end{aligned}$$

so it's not the electric field or polarisability that's determined by the free charges, it's the peculiar combination $\epsilon_0\vec{E} + \vec{P}$ that is. This quantity is called the "electric displacement field" and is denoted by $\vec{D}$, and so we see that in a medium that's not pure vacuum, we have to modify one of Maxwell's equations to

$$\vec{\nabla}\cdot\vec{D} = \rho$$

where ρ is the free charge density and $\vec{D} = \epsilon_0\vec{E} + \vec{P}$ the electric displacement field.

But this still doesn't solve the basic problem, because even though the above equation gives us $\vec{D}$ if we know ρ, $\vec{E}$ is the physical field that affects charges, and $\vec{E} = (\vec{D} - \vec{P})/\epsilon_0$ does us no good if we don't know the polarisability.

Luckily for us, for most materials under normal conditions, there's a nice relationship between $\vec{P}$ and $\vec{E}$. We fully expect $\vec{P}$ to depend on the electric field; the bound charges will feel forces and reconfigure themselves when put into an electric field, and this will change the electric dipoles of the constituents and thus $\vec{P}$. In principle, this relationship could be very complicated, but provided the electric field isn't too large, and the medium in question isn't too exotic, it's found empirically that the polarisability is proportional to $\vec{E}$. A medium for which this is the case is said to be a "linear" medium, and for such media, we can measure a physical property called its *electric susceptibility* χ_e, defined simply as $\vec{P} = \epsilon_0\chi_e\vec{E}$. (Note that the presence of ϵ_0 in this equation makes χ_e a dimensionless quantity.)

Electric susceptibilities can have just about any value ranging from zero for a total vacuum (there are no bound charges and thus no intrinsic dipoles) through about 10 for graphite up to about 80 for water (and higher still). The only commonality is that electric susceptibility is never negative.

But a far more useful quantity is a material's electric permittivity ϵ. This arises because the equation we have involves $\vec{D}$, not $\vec{P}$. But for a linear material, we see

$$\begin{aligned} \vec{D} &= \epsilon_0\vec{E} + \vec{P} \\ &= \epsilon_0\vec{E} + \epsilon_0\chi_e\vec{E} \\ &= \epsilon\vec{E} \end{aligned}$$

with $\epsilon = (1+\chi_e)\epsilon_0$. This is why the fundamental constant ϵ_0 is the "vacuum permittivity"; when $\chi_e = 0$, $\epsilon = \epsilon_0$. For all other media, the susceptibility is positive, so $\epsilon > \epsilon_0$ for all nonvacuum situations. Thus, $\vec{D} = \epsilon\vec{E}$ is an equivalent way of defining a linear medium.

But note what this gives us when we put it into our modified Maxwell equation:

$$\vec{\nabla} \cdot \left(\epsilon\vec{E}\right) = \rho \;\Rightarrow\; \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon}.$$

So for a linear medium, we start from the usual form of Gauss' law for electric fields, but treat ρ as the free charge density and replace ϵ_0 by the permittivity of the medium under consideration.

One comment before we move on; although the expressions we used in obtaining all of the above (such as the scalar potential of a electric dipole) were based on what we know from electrostatics, including time dependence does not change the basic results, namely, $\vec{\nabla} \cdot \vec{D} = \rho(t, \vec{r})$ with $\vec{D}(t, \vec{r}) = \epsilon_0 \vec{E}(t, \vec{r}) + \vec{P}(t, \vec{r})$ and $\vec{D} = \epsilon \vec{E}$ for a linear medium. Time-dependence *will* require us to do some additional work in the next part...

Right, so now onto the magnetic case. Given some medium, what do we expect to contribute to the total magnetic field $\vec{B}$? The one quantity we assume we know is the free current density $\vec{J}$, and that should certainly have an effect on what the magnetic field is. But there are also the *intrinsic* magnetic dipole moments of the constituents making up the medium. These are much like the intrinsic electric dipole moments; unless the medium is extremely simple, actually being able to calculate what the magnetic dipole moments of each constituent will be extremely difficult (if not impossible), so we use the same idea as we did for the electric case. Namely, if we take a small volume $\mathrm{d}^3\vec{r}'$ located at a position $\vec{r}'$ and measure the magnetic dipole $\mathrm{d}\vec{m}'$ in this volume, we can define the *magnetisation* $\vec{M}$ via

$$\mathrm{d}\vec{m}' = \vec{M}(\vec{r}')\, \mathrm{d}^3\vec{r}'$$

so $\vec{M}$ is a magnetic dipole density in the same way $\vec{P}$ is an electric dipole density. Thus, the total contribution of these tiny intrinsic magnetic dipoles to the vector potential should be

$$\begin{aligned} \vec{A}_m(\vec{r}) &= \frac{\mu_0}{4\pi} \int \frac{\mathrm{d}\vec{m}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \\ &= \frac{\mu_0}{4\pi} \int \frac{\vec{M}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}\, \mathrm{d}^3\vec{r}'. \end{aligned}$$

Following the same ideas as we did in the electric case, we see that this may be rewritten as

$$\begin{aligned}
\vec{A}_m(\vec{r}) &= \frac{\mu_0}{4\pi}\int \vec{M}(\vec{r}') \times \left(\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|}\right) \mathrm{d}^3\vec{r}' \\
&= -\frac{\mu_0}{4\pi}\int \left(\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|}\right) \times \vec{M}(\vec{r}')\, \mathrm{d}^3\vec{r}' \\
&= -\frac{\mu_0}{4\pi}\int \left[\vec{\nabla}' \times \left(\frac{\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) - \frac{\vec{\nabla}' \times \vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right] \mathrm{d}^3\vec{r}'
\end{aligned}$$

The first integral may be rewritten using the vector calculus identity

$$\int_{\mathcal{V}} \vec{\nabla} \times \vec{a}\, \mathrm{d}^3\vec{r} = \oint_{\Sigma} \mathrm{d}\vec{\sigma} \times \vec{a}.$$

where $\vec{a}$ is any vector field and $\mathcal{V}$ is a region of space with boundary Σ. (This identity is relatively well-known, but I might prove it in next week's tutorial anyway.) Using it gives

$$\vec{A}_m(\vec{r}) = -\frac{\mu_0}{4\pi}\oint_{\Sigma} \mathrm{d}\vec{\sigma}' \times \frac{\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|} + \frac{\mu_0}{4\pi}\int \frac{\vec{\nabla}' \times \vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\mathrm{d}^3\vec{r}'$$

where Σ is the boundary of the medium.

The magnetic field from this is $\vec{B}_m = \vec{\nabla} \times \vec{A}_m$, of course, but we want $\vec{\nabla} \times \vec{B}_m$, since that's what will appear in Ampère's law. We know that $\vec{\nabla} \cdot \vec{B}_m = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}_m) - \nabla^2 \vec{A}_m$, and we already showed in lecture that the divergence of the vector potential due to a magnetic dipole is zero, so the first term vanishes. Thus,

$$\begin{aligned}
\vec{\nabla} \times \vec{B}_m(\vec{r}) &= -\nabla^2 \vec{A}_m(\vec{r}) \\
&= -\nabla^2\left[-\frac{\mu_0}{4\pi}\oint_{\Sigma} \mathrm{d}\vec{\sigma}' \times \frac{\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|} + \frac{\mu_0}{4\pi}\int \frac{\vec{\nabla}' \times \vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\mathrm{d}^3\vec{r}'\right] \\
&= -\mu_0 \oint_{\Sigma} \mathrm{d}\vec{\sigma}' \times \vec{M}(\vec{r}')\, \delta^{(3)}(\vec{r}-\vec{r}') + \mu_0 \int \vec{\nabla}' \times \vec{M}(\vec{r}')\, \delta^{(3)}(\vec{r}-\vec{r}')\, \mathrm{d}^3\vec{r}'
\end{aligned}$$

once again using the fact that $(-4\pi|\vec{r}-\vec{r}'|)^{-1}$ is the Green's function for the Laplacian. Now we're in a very similar situation to the one we had in the electric case, and do the same thing: if we're not looking at a point on the surface of the medium, then the first integral vanishes, and since the second is an integral over all space, we get $\vec{\nabla} \times \vec{B}_m = \mu_0 \vec{\nabla} \times \vec{M}$. Now, Ampère's law has the form (curl of magnetic field) = (μ_0 times current density), so we can identify the curl of the magnetisation with a kind of "magnetic moment current" $\vec{J}_m$:

$$\vec{\nabla} \times \vec{M} = \vec{J}_m.$$

Now, just to be clear, this is *not* a current which arises from moving charges. For example, we know an electron has an intrinsic magnetic dipole moment, but this isn't due to any charges moving around, it's a natural property of the electron. However, we can say that there's an *effective* current that gives the same moment and call it $\vec{J}_m$. (For something like a classical analogy, think of the Coriolis force. It's a fictitious force due to us being in a rotating frame of reference, but for meteorological calculations, we can treat it as real to predict weather patterns and the like.)

So with this interpretation of $\vec{J}_m$, we could propose that this means that the total current density is $\vec{J}_{tot} = \vec{J}_m + \vec{J}$ and put this into our magnetostatic equation $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_{tot}$ much like we did in the electric case. Thus, the free current density is

$$\begin{aligned} \vec{J} &= \vec{J}_{tot} - \vec{J}_m \\ &= \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \vec{\nabla} \times \vec{M} \\ &= \vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{B} - \vec{M} \right) \end{aligned}$$

so we see that it's not $\vec{B}$ or $\vec{M}$ that's determined by the free current, it's the combination $\vec{H} = \vec{B}/\mu_0 - \vec{M}$ that is. This quantity is known as the "magnetic intensity field" and we see that it's what appears in our second modified (static) Maxwell equation:

$$\vec{\nabla} \times \vec{H} = \vec{J}.$$

Before we see how to properly include time dependence, let's look at the magnetisation and magnetic intensity in a bit more detail: it's $\vec{B}$ we want, but like in the electric case, it's not given directly by $\vec{J}$. We only get $\vec{H}$, and since $\vec{B} = \mu_0(\vec{H} + \vec{M})$, we need to know $\vec{M}$. But we fully expect the magnetisation to depend on $\vec{B}$, since an external magnetic field will cause the intrinsic dipoles in the medium to orient themselves in one or another particular direction. Precisely *how* $\vec{M}$ changes with $\vec{B}$ depends on the substance we look at.

The first classification we need is those substances which can have a nonzero magnetisation even when there is no external magnetic field. These are *ferromagnets* and are the materials permanent magnets are made out of, with the three most famous being iron, nickel and cobalt. (Note that ferromagnets don't *have* to be magnetised; it's just that they *can* be magnetised. There's plenty of chunks of iron in the world that don't stick to your fridge door.) A magnetised ferromagnet will have a magnetisation which is largely independent of $\vec{B}$ and thus needs to be determined in some other way (via empirical evidence or some statistical-mechanical model of its microscopic structure).

However, if we are considering a medium which is not already magnetised, then we're in luck, because many of these materials are also linear in a similar way to we saw in the electric case, i.e. their magnetisation is *proportional* to $\vec{H}$ (at least for small fields). For these materials, we can define the magnetic susceptibility χ_m as

$$\vec{M} = \chi_m \vec{H}$$

and since $\vec{M}$ and $\vec{H}$ have the same units, χ_m is dimensionless.

These magnetic susceptibilities differ from electric susceptibilities in two main ways: for most materials at reasonable temperatures and pressures, they all tend to be pretty small in magnitude (10^{-4} is at the upper end) and they can be of either sign: positive χ_m indicates the intrinsic moments of the medium tend to line up with $\vec{H}$ and are called "paramagnetic", and negative χ_M means the moments tend to oppose the magnetic intensity and are "diamagnetic".

But $\vec{M} \propto \vec{H}$ also means $\vec{B} \propto \vec{H}$:

$$\begin{aligned} \vec{B} &= \mu_0 \left(\vec{H} + \vec{M} \right) \\ &= \mu_0 (1 + \chi_m) \vec{H} \end{aligned}$$

so if we define the magnetic permeability μ of a linear medium by $\mu = \mu_0(1 + \chi_m)$, we see $\vec{H} = \vec{B}/\mu$. Again, there are no bound or intrinsic moments in the vacuum, so $\chi_m = 0$ in this case and this is why the fundamental quantity μ_0 is called the vacuum permeability. A paramagnet will have $\mu > \mu_0$ and a diamagnet will have $\mu < \mu_0$.

So for a linear magnetic material, our new static Maxwell equation gives

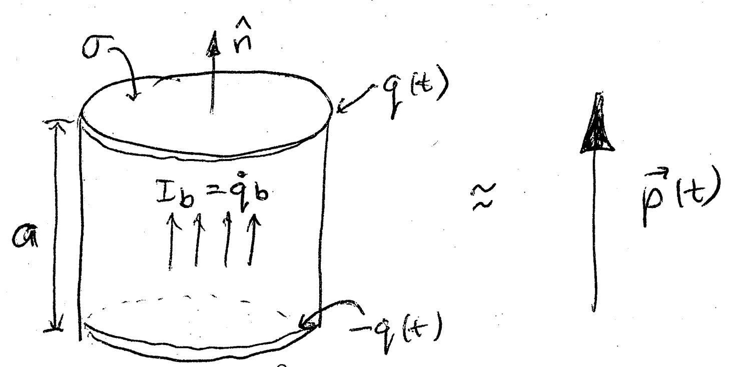
$$\vec{\nabla} \times \left(\frac{\vec{B}}{\mu} \right) = \vec{J} \;\Rightarrow\; \vec{\nabla} \times \vec{B} = \mu \vec{J}$$

which is just the static Ampère law with the replacements $\vec{J}_{tot} \to \vec{J}$ and $\mu_0 \to \mu$ much as we saw $\rho_{tot} \to \rho$ and $\epsilon_0 \to \epsilon$ for a linear electric medium.

So we showed that $\vec{\nabla} \times \vec{H} = \vec{J}$, and this would be the starting point for any *static* situation where we have a medium of some sort. But we now want to see what happens when time dependence is included, because that's where all the fun stuff is. We stated before that inclusion of time dependence doesn't change the argument for the electric case, and this is because electric dipoles depend only on the location, and not the movement, of electric charges. But that's not the case for magnetic dipoles: moving charges very definitely do influence what the magnetic dipole will be.

So what moving charges do we have? $\vec{J}$ includes all info about the movement of free charges, $\vec{J}_m$ is an effective current which tells us which fictitious moving charges would give us the intrinsic moments of our constituents, but what hasn't been included is the movement of the *bound* charges. Unlike the imaginary charges in $\vec{J}_m$, these charges are real and in shifting position within the medium, they will obviously move and create a *bound* current density $\vec{J}_b$. Thus, we expect $\vec{J}_{tot} = \vec{J} + \vec{J}_m + \vec{J}_b$ to be the total current density appearing in the time-dependent version of Ampère's law.

How do we determine this bound current? Here's an extremely simplified argument that gives the flavour of the full one as well as the result we need: let's take one of our constituent particles and look at its electric dipole moment. We can think of modelling it as two charges q and $-q$ separated by a distance a pointing in a direction $\hat{n}$, so $\vec{p} = qa\hat{n}$, as depicted below:



Now, q is a bound charge and so cannot leave the particle, but the charge distribution within the particle can change with time. If we make the simplifying assumption that only the magnitude, and not direction, of the dipole changes, then we have $\vec{p}(t) = q(t)a\hat{n}$. The time derivative of this is $\dot{q}a\hat{n}$, but $\dot{q} = I_b$, a bound current flowing within the particle from one end of the dipole to the other. Now, if we think of this particle having a cross-sectional area (normal to $\hat{n}$) of σ, then $J_b = I_b/\sigma$ is the bound current density. Since all flow is along the dipole's direction, $\vec{J_b} = J_b\hat{n}$, so $\dot{\vec{p}} = a\sigma\vec{J_b}$. But $a\sigma$ is the volume occupied by the particle, so $\vec{p}/a\sigma$ is the electric dipole density, i.e. the polarisability. And the bound current density is the time-derivative of this:

$$\vec{J_b} = \frac{\partial\vec{P}}{\partial t}.$$

So now let's see what the expression for the free current density is: recall that when we include time-dependence, Ampère's law is

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}_{tot} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t},$$

so

$$\begin{aligned}\vec{J} &= \vec{J}_{tot} - \vec{J}_m - \vec{J}_b \\ &= \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t} - \vec{\nabla}\times\vec{M} - \frac{\partial\vec{P}}{\partial t} \\ &= \vec{\nabla}\times\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right) - \frac{\partial}{\partial t}\left(\epsilon_0\vec{E} + \vec{P}\right)\end{aligned}$$

or

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}.$$

Now we have to come to the other two Maxwell equations, but they don't change at all! Why? Well, being in a medium only has an effect on what *sources* we have to use, which is why we have to talk about free charges, bound currents, et cetera. But two of Maxwell's equations – Gauss' law for magnetic fields and Faraday's law of induction – don't involve the sources at all, and so therefore we do not expect them to change one bit, and thus

$$\vec{\nabla}\cdot\vec{B} = 0, \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

hold in *all* cases.

Finally, there's the continuity equation which imposes charge conservation. However, by assumption, free charges remain free and never get bound to any constituents of the medium and bound charges remain bound and never escape from the atom/molecule they're stuck in. Thus, free electric charge and bound electric charge are separately conserved. meaning $\partial\rho/\partial t + \vec{\nabla}\cdot\vec{J} = 0$ even when ρ and $\vec{J}$ refer to only the free (and not the total) densities. Note that the conservation of bound charge is automatic, since $\rho_b = -\vec{\nabla}\cdot\vec{P}$ and $\vec{J}_b = \partial\vec{P}/\partial t$ and so $\partial\rho_b/\partial t + \vec{\nabla}\cdot\vec{J}_b = 0$ follows immediately. (Also note that since $\vec{J}_m$ is a fictitious current density, there is no associated charge density ρ_m and since $\vec{J}_m = \vec{\nabla}\times\vec{M}$ implies $\vec{\nabla}\cdot\vec{J}_m = 0$, $\partial\rho_m/\partial t + \vec{\nabla}\cdot\vec{J}_m = 0$ is consistent with $\rho_m = 0$.)

So, to summarise all of this: suppose we are in a medium characterised by a polarisability $\vec{P}(t,\vec{r})$ and a magnetisation $\vec{M}(t,\vec{r})$. If the free charge density $\rho(t,\vec{r})$ and free current density $\vec{J}(t,\vec{r})$ are given and satisfy

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} \;=\; 0,$$

then Maxwell's equations determining the electric field $\vec{E}(t,\vec{r})$ and magnetic field $\vec{B}(t,\vec{r})$ are

$$\begin{aligned}\vec{\nabla}\cdot\vec{D} = \rho, &\qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t},\\ \vec{\nabla}\cdot\vec{B} = 0, &\qquad \vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t},\end{aligned}$$

where $\vec{D} = \epsilon_0\vec{E} + \vec{P}$ is the electric displacement field and $\vec{H} = \vec{B}/\mu_0 - \vec{M}$ is the magnetic intensity field.

Furthermore, If the medium is linear with permittivity ϵ and permeability μ, then $\vec{D} = \epsilon\vec{E}$ and $\vec{H} = \vec{B}/\mu$ and the above equations become

$$\begin{aligned}\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon}, &\qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t},\\ \vec{\nabla}\cdot\vec{B} = 0, &\qquad \vec{\nabla}\times\vec{B} = \mu\vec{J} + \mu\epsilon\frac{\partial\vec{E}}{\partial t},\end{aligned}$$

And it's these equations we'll start with in the next lecture.