Fourier Transforms

You are familiar with Fourier series for a function on an interval $-T < t < T$.

$$f(t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi t}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{T}\right).$$

where the constants a_n and b_n are determined using orthogonality of the trigonometric functions for positive integers n and n',

$$\int_{-\pi}^{\pi} \cos(n\theta)\cos(n'\theta)d\theta = \pi\delta_{nn'} \qquad \int_{-\pi}^{\pi} \sin(n\theta)\sin(n'\theta)d\theta = \pi\delta_{nn'}$$

$$\int_{-\pi}^{\pi} \sin(n\theta)\cos(n'\theta)d\theta = 0,$$

to be

$$a_n = \frac{1}{T}\int_{-T}^{T} f(t)\cos\left(\frac{n\pi t}{T}\right)dt, \qquad b_n = \frac{1}{T}\int_{-T}^{T} f(t)\sin\left(\frac{n\pi t}{T}\right)dt$$

for positive n, while

$$a_0 = \frac{1}{2T}\int_{-T}^{T} f(t)dt.$$

In the Fourier series $\frac{n\pi}{T}$ is a frequency,

$$\omega_n = \frac{n\pi}{T},$$

and for large T the ω_n are close together for successive n, approaching a continuous variable ω as $T \to \infty$. Define

$$d\omega = \frac{\pi}{T}$$

and the sums go over to Riemann integrals as $T \to \infty$

$$f(t) = \int_0^{\infty} \Big(a(\omega)\cos(\omega t) + b(\omega)\sin(\omega t)\Big)d\omega$$

where

$$a(\omega_n) = \frac{T}{\pi}a_n = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega_n t)dt$$

$$b(\omega_n) = \frac{T}{\pi}b_n = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega_n t)dt.$$

It is conventional to define the *cosine transform* $\tilde{f}_c(\omega)$ and the *sine transform* $\tilde{f}_s(\omega)$ of $f(t)$ as

$$\tilde{f}_c(\omega) := \int_0^{\infty} f(t)\cos(\omega t)dt$$

$$\tilde{f}_s(\omega) := \int_0^{\infty} f(t)\sin(\omega t)dt.$$

These integrals certainly exist if $\int_0^\infty |f(t)|dt$ exists and is finite. The are examples of a class of functions called *integral transforms* where, given a function $f(t)$, we construct a new function $\tilde{f}(\omega)$ as an integral

$$\tilde{f}(\omega) := \int_0^\infty f(t)K(\omega, t)dt$$

where $K(\omega, t)$ is called the *kernel* of the integral transform.

If $f(-t) = f(t)$ is an even function then $\tilde{f}_c(\omega) = \frac{\pi}{2}a(\omega)$ and if $f(-t) = f(t)$ is an odd function then $\tilde{f}_s(\omega) = \frac{\pi}{2}b(\omega)$.

Another type of integral transform that is very useful in physics when periodic phenomena are under consideration is is the *Fourier transform*,

$$\tilde{f}(\omega) = \int_{-\infty}^\infty f(t)e^{i\omega t}dt. \tag{1}$$

Specifying $\tilde{f}(\omega)$ is completely equivalent to specifying $f(t)$ because

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \tilde{f}(\omega)e^{-i\omega t}d\omega, \tag{2}$$

as we shall now show.

Using the definition of $\tilde{f}(\omega)$ in the right hand side of (2) gives

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^\infty \tilde{f}(\omega)e^{-i\omega t}d\omega =& \frac{1}{2\pi}\int_{-\infty}^\infty \left(\int_{-\infty}^\infty f(t')e^{i\omega t'}dt'\right)e^{-i\omega t}d\omega \\ =& \frac{1}{2\pi}\int_{-\infty}^\infty \left(\int_{-\infty}^\infty e^{i\omega(t'-t)}d\omega\right)f(t')dt'.\end{aligned} \tag{3}$$

Now

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\omega(t'-t)}d\omega$$

is a representation of the Dirac δ-function (a heuristic proof is sketched below), so

$$\frac{1}{2\pi}\int_{-\infty}^\infty \tilde{f}(\omega)e^{-i\omega t}d\omega = \int_{-\infty}^\infty \delta(t-t')f(t')dt' = f(t).$$

Hence

$$\boxed{\tilde{f}(\omega) = \int_{-\infty}^\infty f(t)e^{i\omega t}dt \qquad \Leftrightarrow \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \tilde{f}(\omega)e^{-i\omega t}d\omega.}$$

To see that $\frac{1}{2\pi}\int_{-\infty}^\infty e^{i\omega(t'-t)}d\omega$ can be interpreted as an integral of the Dirac δ-function first note that

$$\frac{1}{2\pi}\int_{-\infty}^\infty e^{i\omega(t'-t)}d\omega = 0$$

it $t \neq t'$, because both the real and imaginary parts are just trigonometric functions which oscillate with ω and

$$\int_{-\pi}^{\pi} \cos(\omega(t-t'))d\omega = \int_{-\pi}^{\pi} \sin(\omega(t-t'))d\omega = 0,$$

so the integral over all ω is zero too. Next when $t = t'$, $e^{i\omega(t'-t)} = 1$, so

$$\frac{1}{2\pi}\int_{-\Omega}^{\Omega} e^{i\omega(t'-t)}d\omega = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} d\omega = \frac{\Omega}{\pi} \quad \underset{\Omega\to\infty}{\longrightarrow} \quad \infty,$$

so $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t'-t)}d\omega$ is zero if $t \neq t'$ and diverges if $t = t'$: these are properties of the Dirac δ-function. We can check the normalisation by setting $t' = 0$ and making sure that

$$\int_{-T}^{T} \delta(t)dt = 1$$

for any $T > 0$. Evaluating

$$\begin{aligned}
\int_{-T}^{T}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}d\omega\right)dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-T}^{T} e^{-i\omega t}dt\right)d\omega \\
= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{i}{\omega}e^{-i\omega t}\right]_{-T}^{T} d\omega &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(\omega T)}{\omega}d\omega \\
= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin u}{u}du &= \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin u}{u}du \\
&= 1,
\end{aligned}$$

where $u = \omega T$ and we have used the definite integral

$$\int_{0}^{\infty}\frac{\sin u}{u}du = \frac{\pi}{2}.$$

While this is far from a rigorous proof that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}d\omega = \delta(t)$$

it is certainly consistent with this interpretation. You will be given a more rigorous derivation of this in the Mathematical Methods II course (MP469) after Christmas.