CHAPTER 4: ADDITION OF ANGULAR MOMENTUM

(From Cohen-Tannoudji, Chapter X)

A. INTRODUCTION

1. Total angular momentum in classical mechanics

The total angular momentum of the system of N classical particles is

$$\vec{\mathcal{L}} = \sum_{i=1}^{N} \vec{\mathcal{L}}_i \qquad (4.1)$$

where

$$\vec{\mathcal{L}}_i = \vec{r}_i \times \vec{p}_i \qquad (4.2)$$

Example – 2 particles:

(i) In the absence of external forces (an isolated system) or if all forces are directed to the center then the total angular momentum $\vec{\mathcal{L}}$ is a constant of motion. The angular momenta of each individual particle $\vec{\mathcal{L}}_1$ and $\vec{\mathcal{L}}_2$ are constants of motion only if two particles exert no force on each other.

(ii) If the principle of action and reaction is obeyed between interacting particles i.e. the moment of force exerted by a particle (1) on a particle (2) is exactly compensated by that of the force exerted by (2) on (1) then

$$\frac{\mathrm{d}\vec{\mathcal{L}}}{\mathrm{d}t} = 0 \tag{4.3}$$

In the system of interacting particles, only the total angular momentum is a constant of motion.

2. The importance of total angular momentum in quantum mechanics

(i) Two non-interacting particles in the $\left\{\left|\vec{r}_1, \vec{r}_2\right\rangle\right\}$ representation:

The Hamiltonian is given simply as

$$\hat{H}_0 = \hat{H}_1 + \hat{H}_2 \tag{4.4}$$

where

$$\hat{H}_1 = -\frac{\hbar^2}{2\mu_1}\Delta_1 + V(r_1) \tag{4.5}$$

$$\hat{H}_2 = -\frac{\hbar^2}{2\mu_2}\Delta_2 + V(r_2) \tag{4.6}$$

From the quantum theory of angular momentum and theory of a particle in the central potential ($V(r_1)$ and $V(r_2)$ are central potentials) we know that

$$\left[\hat{\vec{L}}_1, \hat{H}_1\right] = \vec{0} \quad (4.7)$$

$$\left[\hat{\vec{L}}_2, \hat{H}_2\right] = \vec{0} \quad (4.8)$$

so both $\hat{\vec{L}}_1$ and $\hat{\vec{L}}_2$ are constants of motion.

(ii) Two particles interacting via the potential $v\left(\left|\vec{r}_1 - \vec{r}_2\right|\right)$ where

$$\left|\vec{r}_1 - \vec{r}_2\right| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \qquad (4.9)$$

The Hamiltonian is

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + v\left(\left|\vec{r}_1 - \vec{r}_2\right|\right) \qquad (4.10)$$

and we can see that the angular momenta of the individual particles are not constants of motion as they do not commute with the Hamiltonian. For example the commutator $\hat{\vec{L}}_1$ (and likewise $\hat{\vec{L}}_2$)

$$\left[\hat{\vec{L}}_1, \hat{H}\right] = \left[\hat{\vec{L}}_1, v\left(\left|\vec{r}_1 - \vec{r}_2\right|\right)\right] \qquad (4.11)$$

does not vanish; for example for the z-component $\hat{L}_{1z}$ we get

$$\left[\hat{L}_{1z}, \hat{H}\right]|\psi\rangle = \left[\hat{L}_{1z}, v\left(\left|\vec{r}_1 - \vec{r}_2\right|\right)\right]|\psi\rangle = \frac{\hbar}{i}\left(x_1\frac{\partial v}{\partial y_1} - y_1\frac{\partial v}{\partial x_1}\right)|\psi\rangle \neq 0 \qquad (4.12)$$

Total angular momentum operator, in contrast to the individual angular momentum operators, is the constant of motion as we can see below

$$\hat{\vec{L}} = \hat{\vec{L}}_1 + \hat{\vec{L}}_2 \tag{4.13}$$

For example we choose the z-component

$$\left[\hat{L}_z, \hat{H}\right] = \left[\hat{L}_{1z} + \hat{L}_{2z}, \hat{H}\right] \tag{4.14}$$

$$\left[\hat{L}_z, \hat{H}\right] = \left[\hat{L}_{1z} + \hat{L}_{2z}, \hat{H}\right] \tag{4.15}$$

$$= \frac{\hbar}{i}\left(x_1 \frac{\partial v}{\partial y_1} - y_1 \frac{\partial v}{\partial x_1} + x_2 \frac{\partial v}{\partial y_2} - y_2 \frac{\partial v}{\partial x_2}\right) \tag{4.16}$$

As the potential depends on $\left|\vec{r}_1 - \vec{r}_2\right|$

$$\frac{\partial v}{\partial x_1} = v' \frac{\partial \left|\vec{r}_1 - \vec{r}_2\right|}{\partial x_1} = v' \frac{x_1 - x_2}{\left|\vec{r}_1 - \vec{r}_2\right|} \tag{4.17}$$

$$\frac{\partial v}{\partial x_2} = v' \frac{\partial \left|\vec{r}_1 - \vec{r}_2\right|}{\partial x_2} = v' \frac{x_2 - x_1}{\left|\vec{r}_1 - \vec{r}_2\right|} \tag{4.18}$$

(similarly for the y coordinates) and then

$$\begin{aligned}\left[\hat{L}_z, \hat{H}\right] &= \frac{\hbar}{i} \frac{v'}{\left|\vec{r}_1 - \vec{r}_2\right|} \left\{x_1 (y_1 - y_2) - y_1 (x_1 - x_2) + x_2 (y_2 - y_1) - y_2 (x_2 - x_1)\right\} \\ &= 0 \end{aligned} \tag{4.19}$$

Another example – a single particle with spin in a central potential. The commutation relation

$$\left[\hat{\vec{L}}, \hat{H}_0\right] = 0 \tag{4.20}$$

where $\hat{H}_0 = \hat{H}_{orbit} + \hat{H}_{spin}$, and the fact that the three components of the spin $\hat{\vec{S}}$ commute with orbital observables implies that the spin is a constant of motion.

Consider now relativistic corrections, and specifically the spin-orbit coupling. The Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_{S0} \tag{4.21}$$

where

$$\hat{H}_{S0} = \xi(r)\hat{\vec{L}} \cdot \hat{\vec{S}} \tag{4.22}$$

With the spin-orbit interaction added, the operators $\hat{\vec{L}}$ and $\hat{\vec{S}}$ do not longer commute with the Hamiltonian, e.g. for the z components we get

$$\begin{aligned}\left[\hat{S}_z, \hat{H}_{S0}\right] &= \xi(r)\left[\hat{S}_z, \hat{L}_x\hat{S}_x + \hat{L}_y\hat{S}_y + \hat{L}_z\hat{S}_z\right] \\ &= \xi(r)\left(i\hbar\hat{L}_x\hat{S}_y - i\hbar\hat{L}_y\hat{S}_x\right)\end{aligned} \tag{4.23}$$

$$\begin{aligned}\left[\hat{L}_z, \hat{H}_{S0}\right] &= \xi(r)\left[\hat{L}_z, \hat{L}_x\hat{S}_x + \hat{L}_y\hat{S}_y + \hat{L}_z\hat{S}_z\right] \\ &= \xi(r)\left(i\hbar\hat{L}_y\hat{S}_x - i\hbar\hat{L}_x\hat{S}_y\right)\end{aligned} \tag{4.24}$$

However, if we set

$$\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}} \tag{4.25}$$

three components of $\hat{\vec{J}}$ will be constants of motion

$$\left[\hat{J}_z, \hat{H}_{S0}\right] = \left[\hat{L}_z + \hat{S}_z, \hat{H}_{S0}\right] = 0 \tag{4.26}$$

$\hat{\vec{J}}$ is called the total angular momentum of a particle with spin.

The problem of addition of angular momentum:

Consider partial angular momentum operators $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ such that

(i) $\left[\hat{\vec{J}}_1, \hat{\vec{J}}_2\right] = 0;$

(ii) $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ are not constants of motion;

(iii) $\hat{\vec{J}} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2$ commutes with $\hat{H}$.

The basis of the state space composed of the eigenvectors common to

$$\hat{J}_1^2, J_{z1}, \hat{J}_2^2, J_{z2}$$

is the old basis, natural if we consider independent (non-interacting) particles or degrees of freedom.

However in the case of interacting particles described above, we consider

a new basis formed by the eigenvectors of
$\hat{J}^2$, and $\hat{J}_z$
which are constants of motion.

Finding this basis is the problem of addition of two angular momenta $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$.

Advantages of the new basis:

(i) it is simpler to diagonalize $\hat{H}$ in the new basis;

(ii) since $\left[\hat{H}, \hat{J}^2\right] = 0$ and $\left[\hat{H}, \hat{J}_z\right] = 0$, $\hat{H}$ can be broken down into block-diagonal structure corresponding to various sets of eigenvalues of $\hat{J}^2$ and $\hat{J}_z$.

B. ADDITION OF TWO SPIN 1/2'S. ELEMENTARY METHOD

1. Statement of the problem

We will consider two spin 1/2 particles and we will be concerned with their spin degrees of freedom which are characterized by their individual spin operators $\hat{\vec{S}}_1$ for the particle (1) and $\hat{\vec{S}}_2$ for the particle (2).

a. STATE SPACE

The basis is given by the eigenstates of the operators $\left\{\hat{S}_1^2, \hat{S}_2^2, S_{z1}, S_{z2}\right\}$ which form C.S.C.O. (we remark that the operators $\hat{S}_1^2$ and $\hat{S}_2^2$ are multiples of identity, so they can be omitted and we will still have C.S.C.O.):

$$\mathscr{B} = \{|\varepsilon_1, \varepsilon_2\rangle\} = \{|+,+\rangle, |+,-\rangle, |-,+\rangle, |-,-\rangle\} \tag{4.27}$$

such that

$$\hat{S}_1^2 |\varepsilon_1, \varepsilon_2\rangle = \hat{S}_2^2 |\varepsilon_1, \varepsilon_2\rangle = \frac{3}{4}\hbar^2 |\varepsilon_1, \varepsilon_2\rangle \tag{4.28}$$

$$\hat{S}_{1z} |\varepsilon_1, \varepsilon_2\rangle = \varepsilon_1 \frac{\hbar}{2} |\varepsilon_1, \varepsilon_2\rangle \tag{4.29}$$

$$\hat{S}_{2z} |\varepsilon_1, \varepsilon_2\rangle = \varepsilon_2 \frac{\hbar}{2} |\varepsilon_1, \varepsilon_2\rangle \tag{4.30}$$

b. TOTAL SPIN $\hat{\vec{S}}$. COMMUTATION RELATIONS

The total spin is the sum of the spins of the individual particles

$$\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2 \tag{4.31}$$

The components of the total spin satisfy the usual commutation relations, e.g.

$$\begin{aligned}
\left[\hat{S}_x, \hat{S}_y\right] &= \left[\hat{S}_{1x} + \hat{S}_{2x}, \hat{S}_{1y} + \hat{S}_{2y}\right] && (4.32)\\
&= \left[\hat{S}_{1x}, \hat{S}_{1y}\right] + \left[\hat{S}_{2x}, \hat{S}_{2y}\right] && (4.33)\\
&= i\hbar\hat{S}_{1z} + i\hbar\hat{S}_{2z} && (4.34)\\
&= i\hbar\hat{S}_z && (4.35)
\end{aligned}$$

The operator $\hat{S}^2$ can be expressed as

$$\hat{S}^2 = \left(\hat{\vec{S}}_1 + \hat{\vec{S}}_2\right)^2 = \hat{S}_1^2 + \hat{S}_2^2 + 2\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 \tag{4.36}$$

where

$$\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 = \hat{S}_{1x}\hat{S}_{2x} + \hat{S}_{1y}\hat{S}_{2y} + \hat{S}_{1z}\hat{S}_{2z} \tag{4.37}$$

$$= \frac{1}{2}\left(\hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+}\right) + \hat{S}_{1z}\hat{S}_{2z} \tag{4.38}$$

We will use this relation. First, note that since $\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$ we get

$$\left[\hat{S}_z, \hat{S}_1^2\right] = \left[\hat{S}_z, \hat{S}_2^2\right] = 0 \tag{4.39}$$

$$\left[\hat{S}^2, \hat{S}_1^2\right] = \left[\hat{S}^2, \hat{S}_2^2\right] = 0 \tag{4.40}$$

and obviously

$$\left[\hat{S}_z, \hat{S}_{1z}\right] = \left[\hat{S}_z, \hat{S}_{2z}\right] = 0 \tag{4.41}$$

We point out however that $\hat{S}^2$ does NOT commute with either S_{1z} or S_{2z}

$$\begin{aligned}\left[\hat{S}^2, \hat{S}_{1z}\right] &= \left[\hat{S}_1^2 + \hat{S}_2^2 + 2\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2, \hat{S}_{1z}\right] &(4.42)\\ &= 2\left[\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2, \hat{S}_{1z}\right] &(4.43)\\ &= 2\left[\hat{S}_{1x}\hat{S}_{2x} + \hat{S}_{1y}\hat{S}_{2y}, \hat{S}_{1z}\right] &(4.44)\\ &= 2i\hbar\left(-\hat{S}_{1y}\hat{S}_{2x} + \hat{S}_{1x}\hat{S}_{2y}\right) &(4.45)\end{aligned}$$

and similarly

$$\left[\hat{S}^2, \hat{S}_{2z}\right] = -\left[\hat{S}^2, \hat{S}_{1z}\right] \qquad (4.46)$$

which implies that

$$\left[\hat{S}^2, \hat{S}_{1z} + \hat{S}_{2z}\right] = \left[\hat{S}^2, \hat{S}_z\right] = 0 \qquad (4.47)$$

c. THE BASIS CHANGE TO BE PERFORMED

The old basis is given by the common eigenvectors of the operators

$$\left\{\hat{S}_1^2, \hat{S}_2^2, \hat{S}_{1z}, \hat{S}_{2z}\right\}$$

and is given in general by the kets $\{|s_1, s_2; m_1, m_2\rangle\}$. In the case of two spin-$\frac{1}{2}$ particles, the basis is $\left\{\left|\frac{1}{2}, \frac{1}{2}; \pm\frac{1}{2}, \pm\frac{1}{2}\right\rangle\right\}$ or in the simplified notation explicitly using signs of the values of the magnetic quantum numbers $\{|+,+\rangle, |+,-\rangle, |-,+\rangle, |-,-\rangle\}$.

The new C.S.C.O. is given by the operators

$$\hat{S}_1^2, \hat{S}_2^2, \hat{S}^2, \hat{S}_z$$

Adding the spins $\hat{\vec{S}}_1$ and $\hat{\vec{S}}_2$ amounts to constructing the orthonormal system of eigenvectors common to the new C.S.C.O.

The new basis is then given by the kets $\{|s_1, s_2; S, M\rangle\}$.

The new basis vectors $|S, M\rangle$ satisfy the eigenvalue equations

$$\hat{S}_1^2 |S, M\rangle = \hat{S}_2^2 |S, M\rangle = \frac{3}{4}\hbar^2 |S, M\rangle \tag{4.48}$$

$$\hat{S}^2 |S, M\rangle = S(S+1)\hbar^2 |S, M\rangle \tag{4.49}$$

$$\hat{S}_z |S, M\rangle = M\hbar |S, M\rangle \tag{4.50}$$

where $S \geq 0$ is an integer or half-integer and $M = S, S-1, ..., -S+1, -S$.

The elementary method of finding what values S and M actually are is based on calculation and diagonalization of 4×4 matrices.

2. The eigenvalue of $\hat{S}_z$ and their degrees of degeneracy

From the (new) C.S.C.O. = $\left\{\hat{S}_1^2, \hat{S}_2^2, \hat{S}^2, \hat{S}_z\right\}$, the first two operators $\hat{\vec{S}}_1$ and $\hat{\vec{S}}_2$ are easy to deal with as $\hat{S}_1^2 |S, M\rangle = \hat{S}_2^2 |S, M\rangle = \frac{3}{4}\hbar^2 |S, M\rangle$. Since $\left[\hat{S}_z, \hat{S}_1^2\right] = \left[\hat{S}_z, \hat{S}_2^2\right] = \left[\hat{S}_z, \hat{S}^2\right] = 0$, the (old) basis vectors $|\varepsilon_1, \varepsilon_2\rangle$ are automatically the eigenvectors of $\hat{S}_z$

$$\hat{S}_z |\varepsilon_1, \varepsilon_2\rangle = \left(\hat{S}_{1z} + \hat{S}_{2z}\right) |\varepsilon_1, \varepsilon_2\rangle \;\;=\;\; \frac{1}{2}(\varepsilon_1 \cdot 1 + \varepsilon_2 \cdot 1) |\varepsilon_1, \varepsilon_2\rangle \hbar \qquad (4.51)$$

with the eigenvalues

$$M\hbar \;\;=\;\; \frac{1}{2}(\varepsilon_1 \cdot 1 + \varepsilon_2 \cdot 1) \hbar \qquad (4.52)$$

where ϵ_1 and ϵ_2 can each be $\pm$. The values of $M = +1$ and $M = -1$, corresponding to the eigenvectors $|+, +\rangle$ and $|-, -\rangle$ respectively, are not degenerate. $M = 0$ is two-fold degenerate: $\{|+, -\rangle, |-, +\rangle\}$.

The operator $\hat{S}_z$ in the matrix representation given by the new basis (C.S.C.O. = $\left\{\hat{S}_1^2, \hat{S}_2^2, \hat{S}^2, \hat{S}_z\right\}$) has the following form

$$\left(\hat{S}_z\right) = \hbar \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.53}$$

It is a diagonal matrix with the eigenvalues on the diagonal.

3. Diagonalization of $\hat{S}^2$

a. CALCULATION OF THE MATRIX REPRESENTING $\hat{S}^2$

Our task is now to find and diagonalize the operator $\hat{S}^2$ given in the $|\varepsilon_1, \varepsilon_2\rangle$ basis. We will use the formula

$$\hat{S}^2 = \hat{S}_1^2 + \hat{S}_2^2 + 2\hat{S}_{1z}\hat{S}_{2z} + \hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+} \qquad (4.54)$$

The four vectors $|\varepsilon_1, \varepsilon_2\rangle$ are eigenvectors of $\hat{S}_1^2, \hat{S}_2^2, S_{1z}$ and S_{2z}.

Using

$$\hat{S}^2 = \hat{S}_1^2 + \hat{S}_2^2 + 2\hat{S}_{1z}\hat{S}_{2z} + \hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+}$$

and the rising $\hat{S}_+$ and lowering $\hat{S}_-$ operators onto the basis vectors $|\varepsilon_1, \varepsilon_2\rangle$, i.e. using $\hat{S}_+ |+\rangle = 0, \hat{S}_+ |-\rangle = \hbar |+\rangle, \hat{S}_- |+\rangle = \hbar |-\rangle, \hat{S}_- |-\rangle = 0$, we get

$$\begin{aligned}
\hat{S}^2 |+,+\rangle &= \left(\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2\right)|+,+\rangle + \frac{1}{2}\hbar^2 |+,+\rangle = 2\hbar^2 |+,+\rangle \\
\hat{S}^2 |+,-\rangle &= \left(\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2\right)|+,-\rangle - \frac{1}{2}\hbar^2 |+,-\rangle + \hbar^2 |-,+\rangle = \hbar^2 \left[|+,-\rangle + |-,+\rangle\right] \\
\hat{S}^2 |-,+\rangle &= \left(\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2\right)|-,+\rangle - \frac{1}{2}\hbar^2 |-,+\rangle + \hbar^2 |+,-\rangle = \hbar^2 \left[|-,+\rangle + |+,-\rangle\right] \\
\hat{S}^2 |-,-\rangle &= \left(\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2\right)|-,-\rangle + \frac{1}{2}\hbar^2 |-,-\rangle = 2\hbar^2 |-,-\rangle
\end{aligned}$$

The operator $\hat{S}^2$ in the matrix representation given by the new basis has the following form

$$\left(\hat{S}^2\right) = \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \tag{4.55}$$

It consists of three submatrices from which two are one-dimensional.

b. EIGENVALUES AND EIGENVECTORS OF $\hat{S}^2$

To obtain the eigenvalues and eigenvectors of $\hat{S}^2$ we have to diagonalize the matrix $\left(\hat{S}^2\right)$ given above. We note

(i) Vectors $|+,+\rangle$ and $|-,-\rangle$ are eigenvectors of $\hat{S}^2$ with the eigenvalue $2\hbar^2$;

(ii) In order to obtain the remaining two eigenvalues, we have to diagonalize the 2×2 block of $\left(\hat{S}^2\right)$

$$\hbar^2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{4.56}$$

The corresponding characteristic equation

$$(1 - \lambda)^2 - 1 \quad = \quad 0 \tag{4.57}$$

has two solutions

$$\lambda_1 \quad = \quad 0 \qquad \lambda_2 = 2 \tag{4.58}$$

and thus the two remaining eigenvalues of $\hat{S}^2$ are 0 and $2\hbar^2$.

The eigenvectors are

$$\frac{1}{\sqrt{2}}\left[|+,-\rangle + |-,+\rangle\right] \qquad \text{for the eigenvalue } 2\hbar^2 \tag{4.59}$$

$$\frac{1}{\sqrt{2}}\left[|+,-\rangle - |-,+\rangle\right] \qquad \text{for the eigenvalue } 0 \tag{4.60}$$

4. Results: triplet and singlet

The quantum number S is either 0 or 1:

$S = 0$:

is associated with a single vector which is also an eigenvector of $\hat{S}_z$ with the eigenvalue 0

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left[|+,-\rangle - |-,+\rangle\right] \tag{4.61}$$

This states is called **singlet** state ($2S + 1 = 1$) and is antisymmetric with respect to exchange of the spins.

$S = 1$:

is associated with three vectors with $M = 1, 0, -1$:

$$\begin{cases} |1,1\rangle & = |+,+\rangle \\ |1,0\rangle & = \frac{1}{\sqrt{2}}\left[|+,-\rangle + |-,+\rangle\right] \\ |1,-1\rangle & = |-,-\rangle \end{cases} \tag{4.62}$$

These states are called **triplet** states $(2S + 1 = 3)$. The state $|1, 0\rangle$ is symmetric with spin exchange.

To summarize, the operators $\{\hat{S}^2, \hat{S}_z\}$ constitute C.S.C.O. for a system of two particles whose individual spins $(s_1 = s_2 = 1/2)$, when added up, give the total spin characterized by the eigenvalue $S(S + 1)\hbar^2$ of the operator $\hat{S}^2$ with the values of the azimuthal quantum numbers $S = 0$ (one eigenvector – singlet , labelled by $M = 0$) and $S = 1$ (three eigenvectors – triplet, labelled by $M = +1, 0, -1$).

C. ADDITION OF TWO ARBITRARY ANGULAR MOMENTA. GENERAL METHOD

1. Review of the general theory of angular momentum

Consider an arbitrary system with the state space $\mathcal{E}$ and an angular momentum $\hat{\vec{J}}$.

A standard basis $\{|k, j, m\rangle\}$ of eigenvectors common to $\hat{J}^2$ and $\hat{J}_z$ satisfy the eigenvalue equations

$$\hat{J}^2 |k, j, m\rangle = j(j+1)\hbar^2 |k, j, m\rangle \tag{4.63}$$

$$\hat{J}_z |k, j, m\rangle = m\hbar |k, j, m\rangle \tag{4.64}$$

The raising and lowering operators act on these basis states as

$$\hat{J}_\pm |k, j, m\rangle = \hbar \sqrt{j(j+1) - m(m \pm 1)}\, |k, j, m \pm 1\rangle \tag{4.65}$$

The state space consists of subspaces $\mathcal{E} = \bigoplus_{k,j} \mathcal{E}(k, j)$ which are characterized by a fixed values of k and j. The subspaces $\mathcal{E}(k, j)$ have the following properties:

(i) they are $(2j + 1)$ dimensional;

(ii) they are globally invariant under the action of $\hat{J}^2$, $\hat{J}_z$, $\hat{J}_\pm$ (and any function $F(\hat{\vec{J}})$, that is, they have non-zero elements only inside each of the $\mathcal{E}(k, j)$ subspaces;

(iii) inside each subspace, the matrix elements of the angular momentum $\hat{\vec{J}}$ are independent of k.

2. Statement of the problem

a. STATE SPACE

Consider a physical system formed by the union of two subsystems:

The subsystem (1) has the state space $\mathcal{E}_1$ which is spanned by the basis vectors $\mathcal{B}_1 = \{|k_1, j_1, m_1\rangle\}$ given by the common eigenvectors of $\hat{J}_1^2$ and $\hat{J}_{1z}$:

$$
\begin{aligned}
\hat{J}_1^2\,|k_1, j_1, m_1\rangle &= j_1(j_1+1)\hbar^2\,|k_1, j_1, m_1\rangle && (4.66)\\
\hat{J}_{1z}\,|k_1, j_1, m_1\rangle &= m_1\hbar\,|k_1, j_1, m_1\rangle && (4.67)\\
\hat{J}_{1\pm}\,|k_1, j_1, m_1\rangle &= \hbar\sqrt{j_1(j_1+1) - m_1(m_1 \pm 1)}\,|k_1, j_1, m_1 \pm 1\rangle && (4.68)
\end{aligned}
$$

The subsystem (2) has the state space $\mathcal{E}_2$ which is spanned by the basis vectors $\mathcal{B}_2 = \{|k_2, j_2, m_2\rangle\}$ given by the common eigenvectors of $\hat{J}_2^2$ and $\hat{J}_{2z}$:

$$\begin{aligned}
\hat{J}_2^2 \,|k_2, j_2, m_2\rangle &= j_2(j_2+1)\hbar^2 \,|k_2, j_2, m_2\rangle && (4.69)\\
\hat{J}_{2z} \,|k_2, j_2, m_2\rangle &= m_2\hbar \,|k_2, j_2, m_2\rangle && (4.70)\\
\hat{J}_{2\pm} \,|k_2, j_2, m_2\rangle &= \hbar \sqrt{j_2(j_2+1) - m_2(m_2 \pm 1)}\,|k_2, j_2, m_2 \pm 1\rangle && (4.71)
\end{aligned}$$

The total state space

$$\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \qquad (4.72)$$

is spanned by the basis $\mathcal{B} = \{|k_1, k_2; j_1, j_2; m_1, m_2\rangle\}$:

$$|k_1, k_2; j_1, j_2; m_1, m_2\rangle = |k_1, j_1, m_1\rangle \otimes |k_2, j_2, m_2\rangle \qquad (4.73)$$

Since

$$\mathcal{E}_1 = \bigoplus_{k_1, j_1} \mathcal{E}_1 (k_1, j_1) \tag{4.74}$$

$$\mathcal{E}_2 = \bigoplus_{k_2, j_2} \mathcal{E}_2 (k_2, j_2) \tag{4.75}$$

then

$$\mathcal{E} = \bigoplus_{k_1, j_1, k_2, j_2} \mathcal{E} (k_1, k_2; j_1, j_2) \tag{4.76}$$

where

$$\mathcal{E} (k_1, k_2; j_1, j_2) = \mathcal{E}_1 (k_1, j_1) \otimes \mathcal{E}_2 (k_2, j_2) \tag{4.77}$$

The subspaces $\mathcal{E} (k_1, k_2; j_1, j_2)$ have the dimension $(2j_1+1)(2j_2+1)$ and are globally invariant under the angular momentum operators $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$.

b. TOTAL ANGULAR MOMENTUM. COMMUTATION RELATIONS

The total angular momentum of the system is characterized by the operator

$$\hat{\vec{J}} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2 \tag{4.78}$$

Since $\hat{\vec{J}}_1$ acts on $\mathcal{E}_1$ and $\hat{\vec{J}}_2$ acts on $\mathcal{E}_2$, they commute $\left[\hat{\vec{J}}_1, \hat{\vec{J}}_2\right] = 0$ and thus also

$$\left[\hat{\vec{J}}, \hat{J}_1^2\right] = \left[\hat{\vec{J}}, \hat{J}_2^2\right] = 0 \tag{4.79}$$

Particularly

$$\left[\hat{J}_z, \hat{J}_1^2\right] = \left[\hat{J}_z, \hat{J}_2^2\right] = 0 \tag{4.80}$$

$$\left[\hat{J}^2, \hat{J}_1^2\right] = \left[\hat{J}^2, \hat{J}_2^2\right] = 0 \tag{4.81}$$

and furthermore

$$\left[\hat{J}_{1z}, \hat{J}_z\right] = \left[\hat{J}_{2z}, \hat{J}_z\right] = 0 \tag{4.82}$$

However, since

$$\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + 2\hat{\vec{J}}_1 \cdot \hat{\vec{J}}_2$$
$$= \hat{J}_1^2 + \hat{J}_2^2 + 2\hat{J}_{1z}\hat{J}_{2z} + \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+} \quad (4.83)$$

$$\left[\hat{J}_{1z}, \hat{J}^2\right] \neq 0 \quad (4.84)$$
$$\left[\hat{J}_{2z}, \hat{J}^2\right] \neq 0 \quad (4.85)$$

c. THE BASIS CHANGE TO BE PERFORMED

Old basis $\{|k_1, k_2; j_1, j_2; m_1, m_2\rangle\}$ is well adapted to the study of individual angular momenta $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ of two subsystems as it is formed by simultaneous eigenstates of

$$\hat{J}_1^2, \hat{J}_2^2, \hat{J}_{1z}, \hat{J}_{2z} \tag{4.86}$$

New basis $\{|k; j_1, j_2; J, M\rangle\}$ is formed by eigenvectors of the operators

$$\hat{J}_1^2, \hat{J}_2^2, \hat{J}^2, \hat{J}_z \tag{4.87}$$

and it is well adapted to the study of the total angular momentum of the system.

COMMENT:

$\mathcal{E}(k_1, k_2, j_1, j_2)$ is globally invariant under the action of any operator which is a function of $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ and thus any function of $\hat{\vec{J}}$.

This implies that $\hat{J}^2$ and $\hat{J}_z$ have nonzero matrix elements only between vectors belonging to the same $\mathcal{E}(k_1, k_2, j_1, j_2)$, i.e. the problem reduces to a change of a basis inside each $\mathcal{E}(k_1, k_2, j_1, j_2)$.

Moreover, the matrix elements of the operators $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ (and any function of these including $\hat{J}^2$ and $\hat{J}_z$) in the standard basis are independent of k_1 and k_2.

Therefore the problem of diagonalization of $\hat{J}^2$ and $\hat{J}_z$ is the same inside all the subspaces $\mathcal{E}(k_1, k_2, j_1, j_2)$ which correspond to the same values of j_1 and j_2.

These facts allow us to simplify the notation

$$\mathcal{E}(j_1, j_2) \equiv \mathcal{E}(k_1, k_2; j_1, j_2) \tag{4.88}$$

$$|j_1, j_2; m_1, m_2\rangle \equiv |k_1, k_2; j_1, j_2; m_1, m_2\rangle \tag{4.89}$$

Since $\hat{\vec{J}}$ is an angular momentum, the space $\mathcal{E}(j_1, j_2)$ is globally invariant under the action of any function of $\hat{\vec{J}}$. Consequently, $\mathcal{E}(j_1, j_2)$ is a direct sum of orthogonal subspaces $\mathcal{E}(k, J)$ invariant under the action of $\hat{J}^2$, $\hat{J}_z$, $\hat{J}_+$ and $\hat{J}_-$:

$$\mathcal{E}(j_1, j_2) = \bigoplus_{k,J} \mathcal{E}(k, J) \tag{4.90}$$

Problems:

(i) Given j_1 and j_2, what are the values of J and how many distinct subspaces are associated with them?

(ii) How can the eigenvectors of $\hat{J}^2$ and $\hat{J}_z$ belonging to $\mathcal{E}(j_1, j_2)$ be expanded using the (old) basis $\{|j_1, j_2; m_1, m_2\rangle\}$.

3. Eigenvalues of $\hat{J}^2$ and $\hat{J}_z$

a. SPECIAL CASE OF TWO SPIN 1/2'S

The state space:

$$\mathcal{E}(j_1, j_2) = \mathcal{E}(1/2, 1/2) = \bigoplus_{k,S} \mathcal{E}(k, S) = \mathcal{E}(k, 0) \oplus \mathcal{E}(k, 1) \tag{4.91}$$

where $\mathcal{E}(k, 0)$ is one-dimensional space with $M = 0$ and $\mathcal{E}(k, 1)$ is three-dimensional space whose basis elements are characterized by $M = +1, 0, -1$.

b. THE EIGENVALUES OF $\hat{J}_z$ AND THEIR DEGREES OF DEGENERACY

The space $\mathcal{E}(j_1, j_2)$ has the dimension $(2j_1 + 1)(2j_2 + 1)$.

Assume that j_1 and j_2 are labelled such that

$$j_1 \quad \geq \quad j_2 \tag{4.92}$$

The vectors $|\, j_1, j_2; m_1, m_2\rangle$ are already eigenstates of $\hat{J}_z$:

$$\begin{aligned} \hat{J}_z \,|\, j_1, j_2; m_1, m_2\rangle &= \left(\hat{J}_{1z} + \hat{J}_{2z}\right) |\, j_1, j_2; m_1, m_2\rangle \\ &= (m_1 + m_2)\, \hbar \,|\, j_1, j_2; m_1, m_2\rangle \\ &= M\hbar \,|\, j_1, j_2; m_1, m_2\rangle \end{aligned} \tag{4.93}$$

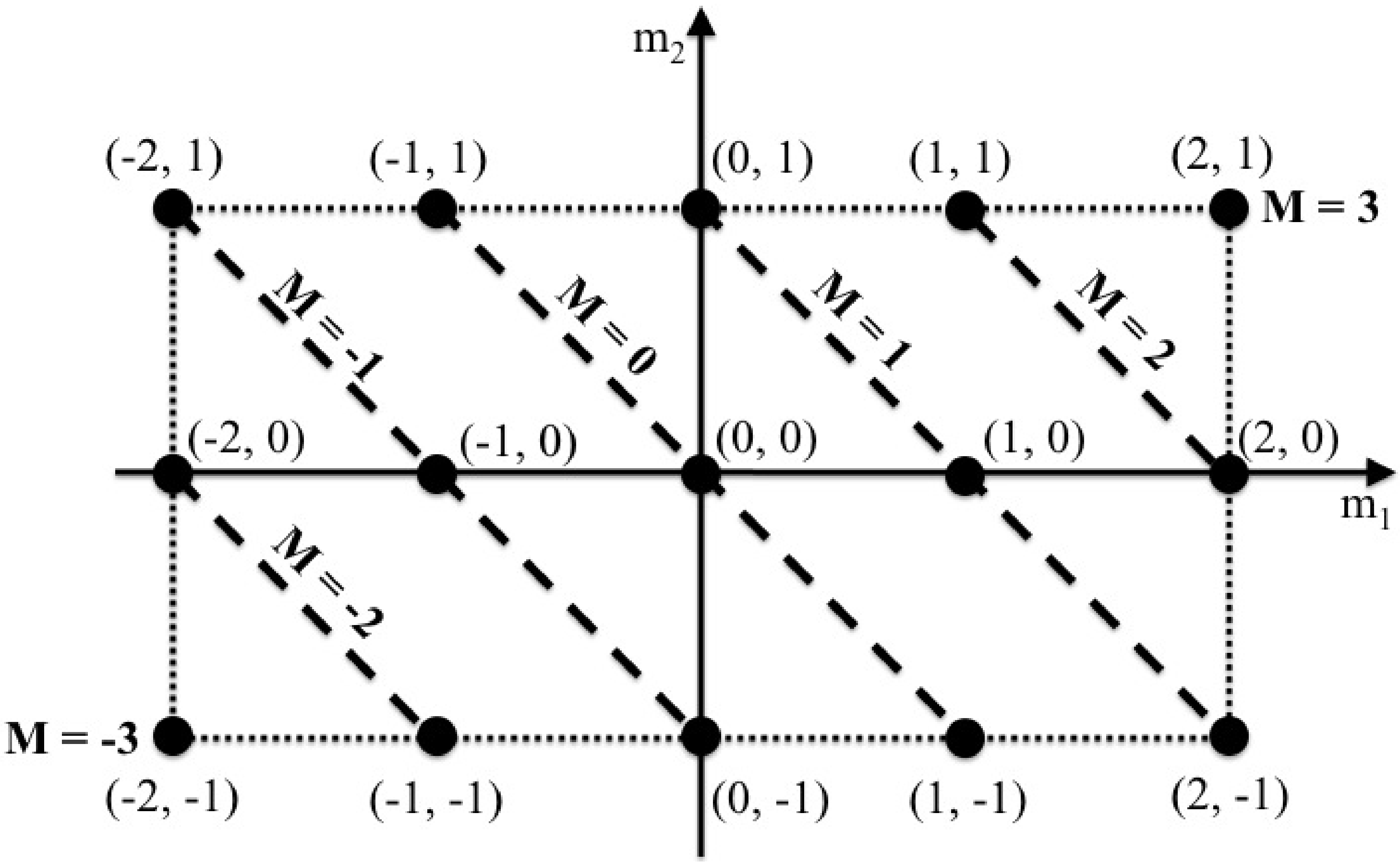
where $M = m_1 + m_2$ has the values

$$j_1 + j_2, j_1 + j_2 - 1, j_1 + j_2 - 2, \ldots, -(j_1 + j_2) \tag{4.94}$$

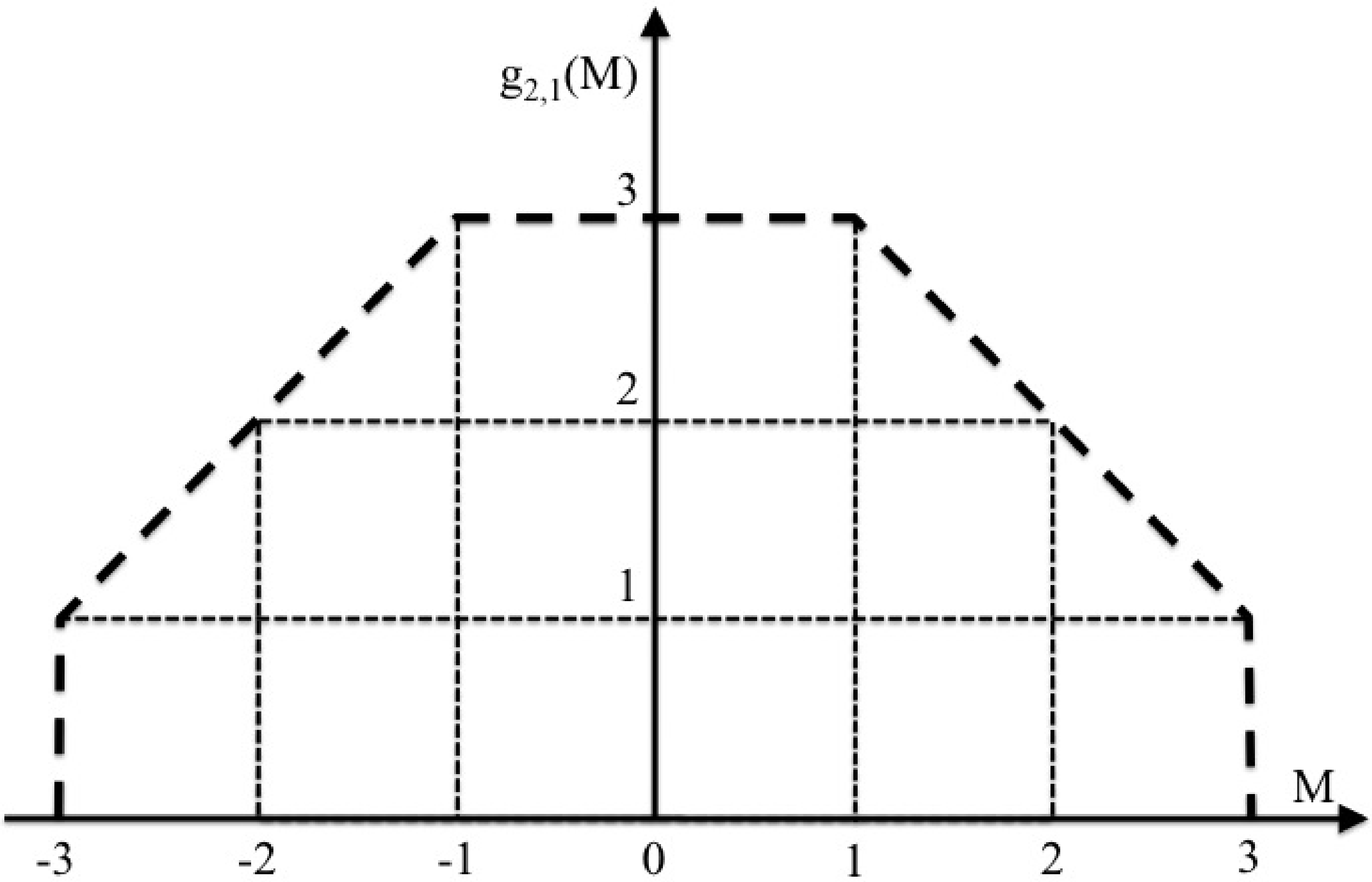
Degree of degeneracy – $g_{j_1,j_2}(M)$:
Consider various values of M

$M = j_1 + j_2$ is non-degenerate, that is, $g_{j_1,j_2}(j_1 + j_2) = 1$.
$M = j_1 + j_2 - 1$ is two-fold degenerate, $g_{j_1,j_2}(j_1 + j_2 - 1) = 2$.
... g_{j_1,j_2} increases by one when M decreases by one
$M = j_1 - j_2$ has maximal degree of degeneracy $g_{j_1,j_2}(j_1 - j_2) = 2j_2 + 1$.
... $g_{j_1,j_2}(M) = 2j_2 + 1$ for $-(j_1 - j_2) \le M \le j_1 - j_2$
$M = -(j_1 - j_2)$ has maximal degree of degeneracy $g_{j_1,j_2}(-j_1 + j_2) = 2j_2 + 1$.
... g_{j_1,j_2} decreases by one when M decreases by one
$M = -(j_1 + j_2) + 1$ is two-fold degenerate, $g_{j_1,j_2}(-j_1 - j_2 + 1) = 2$.
$M = -(j_1 + j_2)$ is non-degenerate, $g_{j_1,j_2}(-j_1 - j_2) = 1$.

Example: $j_1 = 2$ and $j_2 = 1$



m_2
(-2, 1)
(-1, 1)
(0, 1)
(1, 1)
(2, 1)
M = 3
M = -1
M = 0
M = 1
M = 2
(-2, 0)
(-1, 0)
(0, 0)
(1, 0)
(2, 0)
m_1
M = -2
M = -3
(-2, -1)
(-1, -1)
(0, -1)
(1, -1)
(2, -1)


g2,1(M)
3
2
1
M
-3
-2
-1
0
1
2
3

c. THE EIGENVALUES OF $\hat{J}^2$

Maximal value of M is $(j_1 + j_2)$ and is nondegenerate. This implies that

(i) $J = j_1 + j_2$ is maximal value of J in $\mathcal{E}(j_1, j_2) = \bigoplus_{k,J} \mathcal{E}(k, J)$;

(ii) with this value is associated one invariant subspace of $\mathcal{E}(j_1, j_2)$ and only one (as $M = j_1 + j_2$ exists and is non-degenerate). In this subspace, $\mathcal{E}(J = j_1 + j_2)$, there is one and only one vector which corresponds to $M = j_1 + j_2 - 1$. Since this value of M is 2-fold degenerate, $J = j_1 + j_2 - 1$ also exists and to it corresponds a single invariant subspace $\mathcal{E}(J = j_1 + j_2 - 1)$.

(iii) J is integral if both j_1 and j_2 are integral of both half-integral, and J is half-integral if either only j_1 is half-integral or only j_2 is half-integral.

More generally:

Let $p_{j_1,j_2}(J)$ be the number of subspaces $\mathcal{E}(k,J)$ of $\mathcal{E}(j_1,j_2)$ associated with a given J.

Relation between p_{j_1,j_2} and g_{j_1,j_2}:

For a particular value of M, there is one and only one vector in each subspace $\mathcal{E}(k,J)$ such that $J \geq |M|$. Its degree of degeneracy is

$$\begin{aligned} g_{j_1,j_2}(M) &= p_{j_1,j_2}(J=|M|) + p_{j_1,j_2}(J=|M|+1) \\ &\quad + p_{j_1,j_2}(J=|M|+2) + \ldots \end{aligned} \tag{4.95}$$

which implies that

$$\begin{aligned} p_{j_1,j_2}(J) &= g_{j_1,j_2}(M=J) - g_{j_1,j_2}(M=J+1) \\ &= g_{j_1,j_2}(M=-J) - g_{j_1,j_2}(M=-J-1) \end{aligned} \tag{4.96}$$

To determine the values of J and the number of invariant subspaces $\mathcal{E}(k, J)$

$$\begin{aligned} p_{j_1,j_2}(J) &= 0 \text{ for } J > j_1 + j_2 && (4.97) \\ p_{j_1,j_2}(J = j_1 + j_2) &= g_{j_1,j_2}(M = j_1 + j_2) = 1 && (4.98) \\ p_{j_1,j_2}(J = j_1 + j_2 - 1) &= g_{j_1,j_2}(M = j_1 + j_2 - 1) - g_{j_1,j_2}(M = j_1 + j_2) \\ &= 1 && (4.99) \end{aligned}$$

and by iteration

$$p_{j_1,j_2}(J = j_1 + j_2 - 2) = 1, \quad \ldots, \quad p_{j_1,j_2}(J = j_1 - j_2) = 1 \tag{4.100}$$

and finally

$$p_{j_1,j_2}(J) = 0 \quad \text{for} \quad J < j_1 - j_2 \tag{4.101}$$

Therefore for a fixed j_1 and j_2, i.e. inside a given subspace $\mathcal{E}(j_1, j_2)$, the eigenvalues of $\hat{J}^2$, $J(J+1)\hbar^2$, are determined by

$$J = j_1 + j_2, j_1 + j_2 - 1, j_1 + j_2 - 2, \ldots, |j_1 - j_2| \tag{4.102}$$

The values of the total azimuthal quantum number J are

$$J = j_1 + j_2, j_1 + j_2 - 1, j_1 + j_2 - 2, \ldots, |j_1 - j_2|$$

and with each of these values is associated a single invariant subspace $\mathcal{E}_J$.

This implies that for a fixed value of J and M, compatible with J, there corresponds to them one and only one vector in $\mathcal{E}(j_1, j_2)$:

J determines the subspace $\mathcal{E}_J$, and
M defines the vector in $\mathcal{E}_J$,

that is

$\hat{J}^2$ and $\hat{J}_z$ form C.S.C.O. in $\mathcal{E}(j_1, j_2)$.

4. Common eigenvectors of $\hat{J}^2$ and $\hat{J}_z$

$|J, M\rangle$:

$$\hat{J}^2 \, |J, M\rangle = J(J+1)\hbar^2 \, |J, M\rangle \tag{4.103}$$

$$\hat{J}_z \, |J, M\rangle = M\hbar \, |J, M\rangle \tag{4.104}$$

and also

$$\hat{J}_1^2 \, |J, M\rangle = j_1(j_1+1)\hbar^2 \, |J, M\rangle \tag{4.105}$$

$$\hat{J}_2^2 \, |J, M\rangle = j_2(j_2+1)\hbar^2 \, |J, M\rangle \tag{4.106}$$

a. SPECIAL CASE WITH TWO SPIN 1/2'S

α. *The subspace* $\mathcal{E}(S = 1)$

In $\mathcal{E} = \mathcal{E}(1/2, 1/2)$, the state $|+,+\rangle$ is the only eigenvector of $\hat{S}_z$ with $M = 1$ (which is maximal possible and non-degenerate) and as $\hat{S}^2$ commutes with $\hat{S}_z$, it must be also eigenvector of $\hat{S}^2$ with the eigenvalue $S = 1$:

$$|S = 1, M = 1\rangle = |1, 1\rangle = |+,+\rangle \tag{4.107}$$

To calculate the other states, we use the lowering operator

$$\hat{S}_-|1,1\rangle = \hbar\sqrt{1(1+1) - 1(1-1)}\,|1,0\rangle \tag{4.108}$$

$$= \hbar\sqrt{2}\,|1,0\rangle \tag{4.109}$$

and by inverting this relation and using $\hat{S}_{-} = \hat{S}_{1-} + \hat{S}_{2-}$ we get

$$|1,0\rangle = \frac{1}{\hbar\sqrt{2}}\hat{S}_{-}|1,1\rangle = \frac{1}{\hbar\sqrt{2}}\hat{S}_{-}|+,+\rangle \tag{4.110}$$

$$= \frac{1}{\hbar\sqrt{2}}\left(\hat{S}_{1-} + \hat{S}_{2-}\right)|+,+\rangle \tag{4.111}$$

$$= \frac{1}{\hbar\sqrt{2}}\left[\hbar|-,+\rangle + \hbar|+,-\rangle\right] \tag{4.112}$$

$$= \frac{1}{\sqrt{2}}\left[|-,+\rangle + |+,-\rangle\right] \tag{4.113}$$

We proceed similarly to obtain the state with the next value of M, $|1,-1\rangle$:

$$
\begin{aligned}
|1,-1\rangle &= \frac{1}{\hbar\sqrt{2}}\hat{S}_{-}|1,0\rangle && (4.114)\\
&= \frac{1}{\hbar\sqrt{2}}\left(\hat{S}_{1-}+\hat{S}_{2-}\right)\frac{1}{\sqrt{2}}\left[|-,+\rangle+|+,-\rangle\right] && (4.115)\\
&= \frac{1}{2\hbar}\left[\hbar|-,-\rangle+\hbar|-,-\rangle\right] && (4.116)\\
&= |-,-\rangle && (4.117)
\end{aligned}
$$

β. The subspace $\mathcal{E}(S = 0)$

$|S = 0, M = 0\rangle$ is determined from orthogonality with the three $|1, M\rangle$ vectors: since it is orthogonal to $|1, 1\rangle = |+, +\rangle$ and $|1, -1\rangle = |-, -\rangle$, $|0, 0\rangle$ must be a linear combination of $|+, -\rangle$ and $|-, +\rangle$:

$$|0, 0\rangle \quad = \quad \alpha\,|+, -\rangle + \beta\,|-, +\rangle \tag{4.118}$$

where the coefficients must satisfy the normalization condition

$$\langle 0, 0|0, 0\rangle = |\alpha|^2 + |\beta|^2 \quad = \quad 1 \tag{4.119}$$

The orthogonality of $|0, 0\rangle$ with $|1, 0\rangle$ implies

$$\langle 1, 0|0, 0\rangle \quad = \quad \frac{1}{\sqrt{2}}(\alpha + \beta) = 0 \tag{4.120}$$

Therefore

$$\alpha = -\beta \quad = \quad \frac{1}{\sqrt{2}}\, e^{i\chi} \tag{4.121}$$

where the complex factor is the global phase factor, which we are free to set to 1. This sets the coefficient α to be a real positive number, specifically

$$\alpha \quad = \quad -\beta = \frac{1}{\sqrt{2}} \tag{4.122}$$

Our calculation of the state $|0,0\rangle$ is now completed:

$$|0,0\rangle \quad = \quad \frac{1}{\sqrt{2}}\left[|+,-\rangle - |-,+\rangle\right] \tag{4.123}$$

We calculated all vector $|S, M\rangle$ without diagonalizing $\hat{S}^2$ in the $\{|\epsilon_1, \epsilon_2\rangle\}$ basis.

b. GENERAL CASE (ARBITRARY j_1 AND j_2)

The state space with the fixed values of j_1 and j_2 splits into the direct sum of subspaces with various values of J consistent with the values of j_1 and j_2

$$\mathcal{E}(j_1, j_2) = \mathcal{E}(j_1 + j_2) \oplus \mathcal{E}(j_1 + j_2 - 1) \oplus \ldots \oplus \mathcal{E}(|j_1 - j_2|) \tag{4.124}$$

The problem is to determine $|J, M\rangle$ which span these subspaces.

α. *The subspace* $\mathcal{E}(J = j_1 + j_2)$

The ket $|j_1, j_2; j_1, j_2\rangle$ is the only eigenvector of $\hat{J}_z$ associated with $M = j_1 + j_2$.

Since $\hat{J}^2$ and $\hat{J}_z$ commute and $M = j_1 + j_2$ is non-degenerate, the state $|j_1, j_2; j_1, j_2\rangle$ must be an eigenvector of $\hat{J}^2$ with the eigenvalue $J = j_1 + j_2$:

$$|J = j_1 + j_2, M = j_1 + j_2\rangle = |j_1 + j_2, j_1 + j_2\rangle = |j_1, j_2; j_1, j_2\rangle \tag{4.125}$$

Repeated application of $\hat{J}_-$ will generate the full basis $\{|J, M\rangle\}$ for $J = j_1 + j_2$:

$$\hat{J}_- \, |j_1 + j_2, j_1 + j_2\rangle = \hbar \sqrt{2\,(j_1 + j_2)}\,|j_1 + j_2, j_1 + j_2 - 1\rangle \tag{4.126}$$

which implies

$$|j_1 + j_2, j_1 + j_2 - 1\rangle = \frac{1}{\hbar\sqrt{2\,(j_1 + j_2)}}\hat{J}_- \, |j_1 + j_2, j_1 + j_2\rangle \tag{4.127}$$

$$= \frac{1}{\hbar\sqrt{2\,(j_1 + j_2)}}\left(\hat{J}_{1-} + \hat{J}_{2-}\right)|j_1, j_2; j_1, j_2\rangle \tag{4.128}$$

$$= \frac{1}{\hbar\sqrt{2\,(j_1 + j_2)}}\Big[\hbar\sqrt{2j_1}\,|j_1, j_2; j_1 - 1, j_2\rangle + \hbar\sqrt{2j_2}\,|j_1, j_2; j_1, j_2 - 1\rangle\Big] \tag{4.129}$$

and finally

$$|j_1 + j_2, j_1 + j_2 - 1\rangle = \sqrt{\frac{j_1}{j_1 + j_2}}\,|j_1, j_2; j_1 - 1, j_2\rangle + \sqrt{\frac{j_2}{j_1 + j_2}}\,|j_1, j_2; j_1, j_2 - 1\rangle \tag{4.130}$$

This will allows us to generate $[2(j_1+j_2)+1]$ vectors of $\{|J, M\rangle\}$ basis of $\mathcal{E}(J = j_1+j_2)$.

β. The other subspaces $\mathcal{E}(J)$

Consider the complement of $\mathcal{E}(j_1 + j_2)$ in $\mathcal{E}(j_1, j_2)$, denoted as $\mathcal{S}(j_1 + j_2)$:

$$\mathcal{S}(j_1 + j_2) = \mathcal{E}(j_1 + j_2 - 1) \oplus \mathcal{E}(j_1 + j_2 - 2) \oplus \ldots \oplus \mathcal{E}(|j_1 - j_2|) \quad (4.131)$$

The corresponding degree of degeneracy in $\mathcal{S}(j_1 + j_2)$ equals to

$$g'_{j_1,j_2}(M) = g_{j_1,j_2}(M) - 1 \quad (4.132)$$

where $g_{j_1,j_2}(M)$ is the degree of degeneracy in $\mathcal{E}(j_1, j_2)$

This implies that $M = j_1 + j_2$ no longer exists in $\mathcal{S}(j_1 + j_2)$, the new maximal value of $M = j_1 + j_2 - 1$ and it is non-degenerate.

The corresponding vector must be proportional to $|J = j_1 + j_2 - 1, M = j_1 + j_2 - 1\rangle$.

In the basis $\{|j_1, j_2; m_1, m_2\rangle\}$, the state should be of the form

$$|j_1 + j_2 - 1, j_1 + j_2 - 1\rangle = \alpha\,|j_1, j_2; j_1, j_2 - 1\rangle \tag{4.133}$$

$$+\beta\,|j_1, j_2; j_1 - 1, j_2\rangle \tag{4.134}$$

with

$$|\alpha|^2 + |\beta|^2 = 1 \tag{4.135}$$

and must be orthogonal to $|j_1 + j_2, j_1 + j_2 - 1\rangle$ from $\mathcal{E}(j_1 + j_2)$, that is

$$\alpha\sqrt{\frac{j_2}{j_1 + j_2}} + \beta\sqrt{\frac{j_1}{j_1 + j_2}} = 0 \tag{4.136}$$

α and β is determined up to a phase, choose $\alpha \in \mathbb{R}^+$

and the final result is

$$|j_1 + j_2 - 1, j_1 + j_2 - 1\rangle = \sqrt{\frac{j_1}{j_1 + j_2}}\,|j_1, j_2; j_1, j_2 - 1\rangle \quad (4.137)$$

$$- \sqrt{\frac{j_2}{j_1 + j_2}}\,|j_1, j_2; j_1 - 1, j_2\rangle \quad (4.138)$$

This is the first vector from $\mathcal{E}(J = j_1 + j_2 - 1)$. Repeated application of $\hat{J}_-$ yields $[2(j_1 + j_2 - 1) + 1]$ vectors $|J, M\rangle$ with $M = j_1 + j_2 - 1, j_1 + j_2 - 2, ..., -(j_1 + j_2 - 1)$ which are spanning $\mathcal{E}(J = j_1 + j_2 - 1)$ subspace.

Now consider the space

$$\mathcal{S}(j_1+j_2, j_1+j_2-1) \;=\; \mathcal{E}(j_1+j_2-2)\oplus\ldots\oplus\mathcal{E}(|j_1-j_2|) \tag{4.139}$$

which is the complement of $\mathcal{E}(j_1+j_2)\oplus\mathcal{E}(j_1+j_2-1)$ in $\mathcal{E}(j_1, j_2)$. In this space, the degeneracy of each value of M is again decreased by one with respect to what it was in $\mathcal{S}(j_1+j_2)$. This implies that the maximal value of $M = j_1+j_2-2$ and it is nondegenerate. The corresponding vector in $\mathcal{S}(j_1+j_2, j_1+j_2-1)$ is

$$|J = j_1+j_2-2, M = j_1+j_2-2\rangle \tag{4.140}$$

To calculate this state in the basis $\{j_1, j_2, ; m_1, m_2\}$ it is sufficient to determine a linear combination of three vectors

$$|j_1, j_2, ; j_1, j_2-2\rangle\,, \quad |j_1, j_2, ; j_1-1, j_2-1\rangle\,, \quad |j_1, j_2, ; j_1-2, j_2\rangle \tag{4.141}$$

The coefficients in the linear combination are fixed up to a phase by the condition that it is normalized and orthogonal to the states

$$|j_1+j_2, j_1+j_2-2\rangle\,, \quad |j_1+j_2-1, j_1+j_2-2\rangle\,. \tag{4.142}$$

Repeated application of the lowering operator $\hat{J}_-$ then generates the whole basis of the subspace $\mathcal{E}(j_1 + j_2 - 2)$.

The following diagram summarizes the general construction:

$$
\begin{array}{rcll}
\mathcal{E}(j_1 + j_2) & \Rightarrow & \mathcal{E}(j_1 + j_2 - 1) & \Rightarrow \quad \dots \\
\hline
|j_1 + j_2, j_1 + j_2\rangle & & & \\
\downarrow \hat{J}_- & & & \\
|j_1 + j_2, j_1 + j_2 - 1\rangle & \rightarrow & |j_1 + j_2 - 1, j_1 + j_2 - 1\rangle & \\
\downarrow \hat{J}_- & & & \\
\dots & & \dots & \rightarrow \quad \dots \\
\downarrow \hat{J}_- & & & \\
|j_1 + j_2, -(j_1 + j_2 - 1)\rangle & \rightarrow & |j_1 + j_2 - 1, -(j_1 + j_2 - 1)\rangle & \\
\downarrow \hat{J}_- & & & \\
|j_1 + j_2, -(j_1 + j_2)\rangle & & &
\end{array}
$$

c. CLEBSCH - GORDAN COEFFICIENTS

In each space $\mathcal{E}(j_1, j_2)$, the eigenvectors of $\hat{J}^2$ and $\hat{J}_z$ are linear combination of vectors of the initial basis $|j_1, j_2; m_1, m_2\rangle$:

$$|J, M\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1, j_2; m_1, m_2\rangle \langle j_1, j_2; m_1, m_2|J, M\rangle \quad (4.143)$$

The coefficients of this expansion $\langle j_1, j_2; m_1, m_2|J, M\rangle$ are called **Clebsch-Gordan** coefficients. They are conventionally chosen to be real (phase convention) and they are different from zero only if

$$M = m_1 + m_2 \quad (4.144)$$

$$|j_1 - j_2| \leq J \leq j_1 + j_2 \quad (4.145)$$

which is a reminiscence of the triangle rule.

The inverse is

$$|j_1, j_2; m_1, m_2\rangle = \sum_{J=j_1-j_2}^{j_1+j_2} \sum_{M=-J}^{J} |J, M\rangle \langle J, M|j_1, j_2; m_1, m_2\rangle \qquad (4.146)$$

where

$$\langle J, M|j_1, j_2; m_1, m_2\rangle = \langle j_1, j_2; m_1, m_2|J, M\rangle \qquad (4.147)$$

Complement $\mathrm{A_X}$

EXAMPLES OF ADDITION OF ANGULAR MOMENTUM

Content:

1. Addition of $j_1 = 1$ and $j_2 = 1$

   (a) *The subspace* $\mathcal{E}(J = 2)$

   (b) *The subspace* $\mathcal{E}(J = 1)$

   (c) *The vector* $|J = 0, M = 0\rangle$

2. Addition of an integral orbital angular momentum l and a spin 1/2

(a) *The subspace* $\mathcal{E}(J = l + 1/2)$

(b) *The subspace* $\mathcal{E}(J = l - 1/2)$

1. Addition of $j_1 = 1$ and $j_2 = 1$

The space $\mathcal{E}(1,1)$ is $d = 9$ dimensional and is spanned by the common eigenvectors of $\hat{J}_1^2, \hat{J}_2^2, \hat{J}_{1z}$, and $\hat{J}_{2z}$

$$\{|1,1;m_1,m_2\rangle\}\,, \quad \text{with} \quad m_1, m_2 = 1, 0, -1 \tag{4.148}$$

Our task is to determine the $\{|J,M\rangle\}$ basis of common eigenvectors of $\hat{J}_1^2, \hat{J}_2^2, \hat{J}^2$ and $\hat{J}_z$ where $\hat{\vec{J}}$ is the total angular momentum. From the general theory of addition of the angular momentum we know that $J = j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2|$ which in this specific case gives

$$J \;=\; 2, 1, 0 \tag{4.149}$$

We will now construct the basis in the relevant subspaces, starting with $\mathcal{E}(J = 2)$.

a. THE SUBSPACE $\mathcal{E}(J = 2)$

The ket $|2, 2\rangle$ is simply

$$|2, 2\rangle = |1, 1; 1, 1\rangle \tag{4.150}$$

Applying $\hat{J}_-$ to this state will produce the state $|2, 1\rangle$:

$$|2, 1\rangle = \frac{1}{2\hbar}\hat{J}_- |2, 2\rangle \tag{4.151}$$

$$= \frac{1}{2\hbar}\left(\hat{J}_{1-} + \hat{J}_{2-}\right)|1, 1; 1, 1\rangle \tag{4.152}$$

$$= \frac{1}{2\hbar}\left[\hbar\sqrt{2}\,|1, 1; 0, 1\rangle + \hbar\sqrt{2}\,|1, 1; 1, 0\rangle\right] \tag{4.153}$$

$$= \frac{1}{\sqrt{2}}\left[|1, 1; 1, 0\rangle + |1, 1; 0, 1\rangle\right] \tag{4.154}$$

and applying $\hat{J}_-$ again leads to

$$|2,0\rangle = \frac{1}{\sqrt{6}}\left[|1,1;1,-1\rangle + 2\,|1,1;0,0\rangle + |1,1;-1,1\rangle\right] \tag{4.155}$$

then

$$|2,-1\rangle = \frac{1}{\sqrt{2}}\left[|1,1;0,-1\rangle + |1,1;-1,0\rangle\right] \tag{4.156}$$

and finally:

$$|2,-2\rangle = |1,1;-1,-1\rangle \tag{4.157}$$

b. THE SUBSPACE $\mathcal{E}(J = 1)$

The ket $|1, 1\rangle$ must be a linear combination of the two (old) basis states $|1, 1; 1, 0\rangle$ and $|1, 1; 0, 1\rangle$ (the only ones with $M = 1$):

$$|1, 1\rangle = \alpha\,|1, 1; 1, 0\rangle + \beta\,|1, 1; 0, 1\rangle \tag{4.158}$$

with

$$|\alpha|^2 + |\beta|^2 = 1 \tag{4.159}$$

In order for this state to be orthogonal to the state $|2, 1\rangle$, it is necessary that

$$\alpha + \beta = 0 \tag{4.160}$$

We choose the coefficient α to be real and positive by convention and obtain

$$|1, 1\rangle = \frac{1}{\sqrt{2}}\,[|1, 1; 1, 0\rangle - |1, 1; 0, 1\rangle] \tag{4.161}$$

Repeated application of $\hat{J}_-$ to the state $|1,1\rangle$ will produce the states $|1,0\rangle$ and $|1,-1\rangle$:

$$|1,0\rangle = \frac{1}{\sqrt{2}}[|1,1;1,-1\rangle - |1,1;-1,1\rangle] \quad (4.162)$$

$$|1,-1\rangle = \frac{1}{\sqrt{2}}[|1,1;0,-1\rangle - |1,1;-1,0\rangle] \quad (4.163)$$

We note that although the vector $|1,1;0,0\rangle$ also corresponds to $M = 0$, it does not appear in the state $|1,0\rangle$ above as its Clebsch-Gordan coefficient is zero:

$$\langle 1,1;0,0|1,0\rangle = 0 \quad (4.164)$$

c. THE VECTOR $|J = 0, M = 0\rangle$

We are now left with the calculation of the last vector of the $\{|J, M\rangle\}$ basis which is associated with $J = M = 0$ and which must be a superposition of three (old) basis vectors for which $M = m_1 + m_2 = 0$:

$$|0, 0\rangle = a\,|1, 1; 1, -1\rangle + b\,|1, 1; 0, 0\rangle + c\,|1, 1; -1, 1\rangle \qquad (4.165)$$

with

$$|a|^2 + |b|^2 + |c|^2 = 1 \qquad (4.166)$$

The state must be orthogonal to $|2,0\rangle$ and $|1,0\rangle$, thus yielding two conditions:

$$a + 2b + c = 0 \tag{4.167}$$

$$a - c = 0 \tag{4.168}$$

which imply

$$a = -b \quad = \quad c \tag{4.169}$$

Choosing again the coefficient a real and positive, we get the desired state

$$|0,0\rangle \quad = \quad \frac{1}{\sqrt{3}}\left[|1,1;1,-1\rangle - |1,1;0,0\rangle + |1,1;-1,1\rangle\right] \tag{4.170}$$

COMMENT:
The problem above is physically equivalent to a p^2 configuration of a two-particle system whose wavefunctions representing the initial (i.e. old) basis are

$$\langle \vec{r}_1, \vec{r}_2 | 1, 1; m_1, m_2 \rangle = R_{k_1,1}(r_1) R_{k_2,1}(r_2) Y_1^{m_1}(\theta_1, \phi_1) Y_1^{m_2}(\theta_2, \phi_2) \quad (4.171)$$

Since the radial functions are independent on m_1 and m_2, the linear combinations which give the wavefunctions associated with the kets $|J, M\rangle$ are functions only of angular dependence.

E.g. the wavefunction for the ket $|0, 0\rangle = \frac{1}{\sqrt{3}} [|1, 1; 1, -1\rangle - |1, 1; 0, 0\rangle + |1, 1; -1, 1\rangle]$:

$$\langle \vec{r}_1, \vec{r}_2 | 0, 0 \rangle = R_{k_1,1}(r_1) R_{k_2,1}(r_2) \frac{1}{\sqrt{3}} \Big[Y_1^1(\theta_1, \phi_1) Y_1^{-1}(\theta_2, \phi_2) \quad (4.172)$$

$$- Y_1^0(\theta_1, \phi_1) Y_1^0(\theta_2, \phi_2) + Y_1^{-1}(\theta_1, \phi_1) Y_1^1(\theta_2, \phi_2) \Big] \quad (4.173)$$

2. Addition of an integral orbital angular momentum l and a spin 1/2

The space $\mathcal{E}(l, 1/2)$ is $2(2l+1)$ dimensional and its basis

$$\left\{ \left| l, \frac{1}{2}; m, \epsilon \right\rangle \right\}, \quad \text{with} \quad m = l, l-1, \ldots, -l, \quad \text{and} \quad \epsilon = \pm \tag{4.174}$$

is formed by the eigenstates common to the orbital and spin angular momentum operators $\hat{L}^2, \hat{S}^2, \hat{L}_z$ and $\hat{S}_z$.

Our task is to construct the eigenvectors $|J, M\rangle$ of $\hat{J}^2$ and $\hat{J}_z$ where $\hat{\vec{J}}$ is the total angular momentum of the system:

$$\hat{\vec{J}} \;=\; \hat{\vec{L}} + \hat{\vec{S}} \tag{4.175}$$

where the possible values of the total azimuthal quantum number are

$$J = l + \frac{1}{2}, l - \frac{1}{2} \tag{4.176}$$

a. THE SUBSPACE $\mathcal{E}(J = l + 1/2)$

The $(2l + 2)$ vectors $|J, M\rangle$ spanning the subspace $\mathcal{E}(J = l + 1/2)$ will be obtained by the general method. We start with the state characterized by the maximal value of M

$$\left| l + \frac{1}{2}, l + \frac{1}{2} \right\rangle \;=\; \left| l, \frac{1}{2}; l, + \right\rangle \tag{4.177}$$

and then apply the lowering operator $\hat{J}_-$ to this state to obtain $\left|l+\frac{1}{2}, l-\frac{1}{2}\right\rangle$

$$\left|l+\frac{1}{2}, l-\frac{1}{2}\right\rangle = \frac{1}{\hbar\sqrt{2l+1}}\hat{J}_-\left|l+\frac{1}{2}, l+\frac{1}{2}\right\rangle \tag{4.178}$$

$$= \frac{1}{\hbar\sqrt{2l+1}}\left(\hat{L}_- + \hat{S}_-\right)\left|l, \frac{1}{2}; l, +\right\rangle \tag{4.179}$$

$$= \frac{1}{\hbar\sqrt{2l+1}}\left[\hbar\sqrt{2l}\left|l, \frac{1}{2}; l-1, +\right\rangle + \hbar\left|l, \frac{1}{2}; l, -\right\rangle\right] \tag{4.180}$$

$$= \sqrt{\frac{2l}{2l+1}}\left|l, \frac{1}{2}; l-1, +\right\rangle + \frac{1}{\sqrt{2l+1}}\left|l, \frac{1}{2}; l, -\right\rangle \tag{4.181}$$

The next state, $\left|l+\frac{1}{2}, l-\frac{3}{2}\right\rangle$, is obtained by applying $\hat{J}_-$ again:

$$\left|l+\frac{1}{2}, l-\frac{3}{2}\right\rangle = \frac{1}{\sqrt{2l+1}}\left[\sqrt{2l-1}\left|l, \frac{1}{2}; l-2, +\right\rangle + \sqrt{2}\left|l, \frac{1}{2}; l-1, -\right\rangle\right] \quad (4.182)$$

and obviously continuing this process will generate the whole basis in this subspace, $\mathcal{E}(J = l + 1/2)$.

More generally, the vector $\left|l+\frac{1}{2},M\right\rangle$ will be a linear combination of the only two basis vectors associated with M: $\left|l,\frac{1}{2};M-\frac{1}{2},+\right\rangle$ and $\left|l,\frac{1}{2};M+\frac{1}{2},-\right\rangle$. By comparing the last three formulas we can guess that the linear combination we are looking for is

$$\left|l+\frac{1}{2},M\right\rangle = \frac{1}{\sqrt{2l+1}}\left[\sqrt{l+M+\frac{1}{2}}\left|l,\frac{1}{2};M-\frac{1}{2},+\right\rangle \right. \tag{4.183}$$

$$\left. + \sqrt{l-M+\frac{1}{2}}\left|l,\frac{1}{2};M+\frac{1}{2},-\right\rangle\right] \tag{4.184}$$

with

$$M = l+\frac{1}{2}, l-\frac{1}{2}, l-\frac{3}{2}, \ldots, -l+\frac{1}{2}, -\left(l+\frac{1}{2}\right) \tag{4.185}$$

b. THE SUBSPACE $\mathcal{E}(J = l - 1/2)$

We now try to determine the $2l$ vectors $|J, M\rangle$ associated with $J = l - 1/2$. The state which corresponds to the maximal value of $M = l - 1/2$ in this subspace must be a linear combination of $\left|l, \frac{1}{2}; l-1, +\right\rangle$ and $\left|l, \frac{1}{2}; l, -\right\rangle$ and must be orthogonal to $\left|l + \frac{1}{2}, l - \frac{1}{2}\right\rangle$. Choosing the coefficient for $\left|l, \frac{1}{2}; l, -\right\rangle$ real and positive will result in

$$\left|l - \frac{1}{2}, l - \frac{1}{2}\right\rangle = \frac{1}{\sqrt{2l+1}}\left[\sqrt{2l}\left|l, \frac{1}{2}; l, -\right\rangle - \left|l, \frac{1}{2}; l-1, +\right\rangle\right] \qquad (4.186)$$

The lowering operator allows us to calculate all the other vectors of this subspace. Since there are only two basis vectors with a given value of M, and since the state $\left|l-\frac{1}{2}, M\right\rangle$ is orthogonal to $\left|l+\frac{1}{2}, M\right\rangle$, we get

$$\left|l-\frac{1}{2}, M\right\rangle = \frac{1}{\sqrt{2l+1}}\left[\sqrt{l+M+\frac{1}{2}}\left|l, \frac{1}{2}; M+\frac{1}{2}, -\right\rangle\right. \tag{4.187}$$

$$\left. -\sqrt{l-M+\frac{1}{2}}\left|l, \frac{1}{2}; M-\frac{1}{2}, +\right\rangle\right] \tag{4.188}$$

with

$$M = l-\frac{1}{2}, l-\frac{3}{2}, \ldots, -l+\frac{3}{2}, -\left(l-\frac{1}{2}\right) \tag{4.189}$$

COMMENTS:

(i) The states $|l, 1/2; m, \epsilon\rangle$ of a spin $1/2$ particle can be represented by two-component spinor:

$$\left[\psi_{l,\frac{1}{2};m,+}\right](\vec{r}) = R_{k,l}(r) Y_l^m(\theta,\phi) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (4.190)$$

$$\left[\psi_{l,\frac{1}{2};m,-}\right](\vec{r}) = R_{k,l}(r) Y_l^m(\theta,\phi) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (4.191)$$

The preceding calculation then show that the spinor associated with the states $|J, M\rangle$ can be written

$$\left[\psi_{l+\frac{1}{2},M}\right](\vec{r}) = \frac{1}{\sqrt{2l+1}} R_{k,l}(r) \begin{pmatrix} \sqrt{l+M+\frac{1}{2}}\, Y_l^{M-\frac{1}{2}}(\theta,\phi) \\ \sqrt{l-M+\frac{1}{2}}\, Y_l^{M+\frac{1}{2}}(\theta,\phi) \end{pmatrix} \tag{4.192}$$

$$\left[\psi_{l-\frac{1}{2},M}\right](\vec{r}) = \frac{1}{\sqrt{2l+1}} R_{k,l}(r) \begin{pmatrix} -\sqrt{l-M+\frac{1}{2}}\, Y_l^{M-\frac{1}{2}}(\theta,\phi) \\ \sqrt{l+M+\frac{1}{2}}\, Y_l^{M+\frac{1}{2}}(\theta,\phi) \end{pmatrix} \tag{4.193}$$

(4.194)

(ii) In the particular case of $l = 1$, we get using the general formulas above the basis states of $\mathcal{E}(J = 3/2)$:

$$\left|\frac{3}{2}, \frac{3}{2}\right\rangle = \left|1, \frac{1}{2}; 1, +\right\rangle \tag{4.195}$$

$$\left|\frac{3}{2}, \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|1, \frac{1}{2}; 0, +\right\rangle + \frac{1}{\sqrt{3}}\left|1, \frac{1}{2}; 1, -\right\rangle \tag{4.196}$$

$$\left|\frac{3}{2}, -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|1, \frac{1}{2}; -1, +\right\rangle + \sqrt{\frac{2}{3}}\left|1, \frac{1}{2}; 0, -\right\rangle \tag{4.197}$$

$$\left|\frac{3}{2}, -\frac{3}{2}\right\rangle = \left|1, \frac{1}{2}; -1, -\right\rangle \tag{4.198}$$

and the remaining states which form the new basis of $\mathcal{E}(J = 1/2)$ are

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|1,\frac{1}{2};1,-\right\rangle - \frac{1}{\sqrt{3}}\left|1,\frac{1}{2};0,+\right\rangle \tag{4.199}$$

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|1,\frac{1}{2};0,-\right\rangle - \sqrt{\frac{2}{3}}\left|1,\frac{1}{2};-1,+\right\rangle \tag{4.200}$$