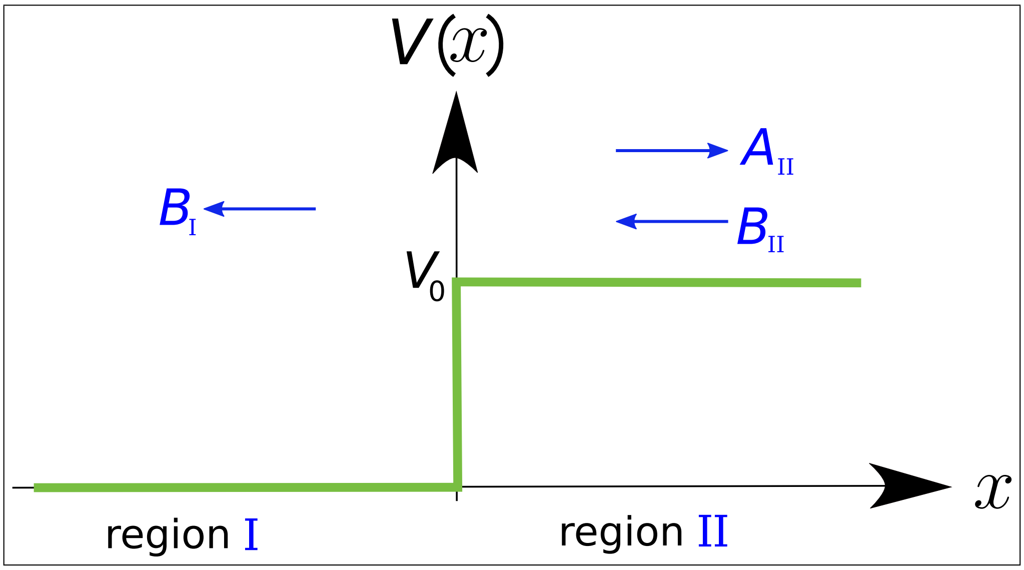
Here we treat the negative potential step, i.e., an incident wave encountering a potential drop.

The scanned handwritten notes for this section has a few 'misprints', so this was typed up.

– – – – – – – – – – – – – – ⋆ – – – – – – – – – – – – – –



We will have the wave incident from the right, and use the same potential used in lecture for the positive potential step (when the wave was incident from the left). The potential is

$$V(x) \;=\; \begin{cases} 0 & \text{when } x < 0 \\ V_0 & \text{when } x > 0 \end{cases}$$

The solution of time-independent Schroedinger equation is a plane wave whenever the potential is constant and the energy E is larger than that constant potential. So the solutions are

$$\psi_{\rm I}(x) = A_{\rm I}e^{ik_1x} + B_{\rm I}e^{-ik_1x} \qquad\qquad k_1 = \sqrt{2mE/\hbar^2}$$
$$\psi_{\rm II}(x) = A_{\rm II}e^{ik_2x} + B_{\rm II}e^{-ik_2x} \qquad\qquad k_2 = \sqrt{2m(E-V_0)/\hbar^2}$$

At the stage when you are dealing with this problem (scattering off a negative potential step), you hopefully know where these solutions come from, and why k_1 and k_2 have the forms that they do. If not, please write down the Schroedinger equation for $V(x) = 0$ and for $V(x) = V_0$, and obtain these solutions.

We want the wave incident from the right; i.e., the leftward wave in region II is the incident wave. Thus the term $B_{\rm II}e^{-ik_2x}$ is interpreted as the incident wave.

If this were a classical (non-quantum) situation, a leftward particle (or left-moving beam of particles) coming from the right would cross the potential step with no problem, and just continue on the other side with higher velocity. This is because the potential corresponds to a leftward force at $x = 0$. In quantum mechanics, we will find that the wave can also get *reflected*. So we will retain the reflection term, i.e., right-moving wave in region II, i.e., the term $A_{\rm II}e^{ik_2x}$. We also need to retain the transmitted term (left-moving wave in region I), i.e., the term $B_{\rm I}e^{-ik_1x}$. There is however no need to keep the right-moving wave in region I, $A_{\rm I}e^{ik_1x}$. Since we are inquiring about the scattering of particles incident from the positive-x side, the $A_{\rm I}$ term does not correspond to anything physically relevant. So let us set $A_{\rm I} = 0$.

Thus we consider solutions of the form

$$\psi_{\rm I}(x) = B_{\rm I}e^{-ik_1x} \;; \qquad \psi_{\rm II}(x) = A_{\rm II}e^{ik_2x} + B_{\rm II}e^{-ik_2x} \,.$$

The reflection coefficient is

$$R = \frac{\text{reflected current density}}{\text{incident current density}} = \frac{|A_{\rm II}|^2k_2}{|B_{\rm II}|^2k_2} = \frac{|A_{\rm II}|^2}{|B_{\rm II}|^2} = \left|\frac{A_{\rm II}}{B_{\rm II}}\right|^2$$

and the transmission coefficient is

$$T = \frac{\text{transmitted current density}}{\text{incident current density}} = \frac{|B_{\rm I}|^2k_1}{|B_{\rm II}|^2k_2} = \left|\frac{B_{\rm I}}{B_{\rm II}}\right|^2 \frac{k_1}{k_2}.$$

So all we have to do is calculate $A_{\rm II}/B_{\rm II}$ and $B_{\rm I}/B_{\rm II}$. We can do that using the boundary conditions. But it's easier to first set $B_{\rm II} = 1$, because we are only interested in the fraction that is transmitted or reflected.

So we set $B_{\rm II} = 1$ and the solutions are:

$$\psi_{\rm I}(x) = B_{\rm I}e^{-ik_1x} \;; \qquad \psi_{\rm II}(x) = A_{\rm II}e^{ik_2x} + e^{-ik_2x} \,. \tag{1}$$

The boundary conditions are

$$\psi_{\rm I}(0) = \psi_{\rm I}(0) \;; \quad \psi'_{\rm I}(0) = \psi'_{\rm I}(0) \,.$$

Using Eq. (1), these two conditions give

$$B_{\rm I} = A_{\rm II} + 1 \;; \quad B_{\rm I}(-ik_1) = A_{\rm II}(ik_2) + 1(-ik_2) \,.$$

Two linear equations for two variables! Very simple. We can solve for the two variables $A_{\rm II}$ and $B_{\rm I}$. The results are

$$A_{\rm II} = \frac{-k_1 + k_2}{k_1 + k_2} \;; \quad B_{\rm I} = \frac{2k_2}{k_1 + k_2} \,.$$

Therefore

$$R = |A_{\rm II}|^2 = \frac{(k_1-k_2)^2}{(k_1+k_2)^2} \; ; \quad T = |B_{\rm I}|^2 \, \frac{k_1}{k_2} = \frac{(2k^2)^2}{(k_1+k_2)^2} \frac{k_1}{k_2} = \frac{4k_1k_2}{(k_1+k_2)^2} \; .$$

The reflection and transmission coefficients happen to be given by the same expressions as what we had for the positive potential step.

You probably want to check that they have reasonable properties, e.g.,

$$R+T=1 \; ; \qquad \lim_{V_0\to 0} T = 1 \; ; \qquad \lim_{V_0\to 0} R = 0 \; .$$

Please make sure that you know physically why you expect these properties. Also, please make sure that you are able to obtain these from the expressions for T and R.