Homogeneous linear equations with constant coefficients

All solutions of the homogeneous linear higher-order DE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

where the coefficients $a_i, i = 0, 1..., n$ are real constants and $a_n \neq 0$ are either exponential functions or are constructed out of exponential functions.

Recall: the solution of the linear first-order DE $y' + ay = 0$, where a is a constant, has an exponential solution $y = c_1 e^{-ax}$ on $(-\infty, \infty)$.

Auxiliary equation

We focus on the second-order equation

$$ay'' + by' + cy = 0 \tag{14}$$

If we try a solution $y = e^{mx}$, the equation above becomes

$$am^2 e^{mx} + bme^{mx} + ce^{mx} = 0 \quad \text{or} \quad e^{mx}(am^2 + bm + c) = 0$$

Since e^{mx} is never zero for real values of x, the exponential function can satisfy the DE (14) only if m is a root of the quadratic equation

$$am^2 + bm + c = 0$$

which is called the **auxiliary equation**.

Since the roots of the auxiliary equation are

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$
$$m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

there will be three forms of general solution of (14):

- m_1 and m_2 are real and distinct ($b^2 - 4ac > 0$),
- m_1 and m_2 are real and equal ($b^2 - 4ac = 0$), and
- m_1 and m_2 are conjugate complex numbers ($b^2 - 4ac < 0$)

Distinct real roots

We have two solutions $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ which are linearly independent on $(-\infty, \infty)$ and thus form a fundamental set of solutions.

The general solution is on this interval

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Repeated roots

When $m_1 = m_2$ we get only one exponential solution $y_1 = e^{m_1 x}$ where $m_1 = -b/2a$ ($b^2 - 4ac = 0$ in the expression for the roots of the quadratic equation).

The second solution can be found by reduction of order:

$$y_2 = e^{m_1 x} \int \frac{e^{2m_1 x}}{e^{2m_1 x}} dx = e^{m_1 x} \int dx = xe^{m_1 x}$$

where we used $-P(x) = -b/a = 2m_1$.

$ay'' + by' + cy = 0$

The generals solution is

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

Conjugate complex roots

We can write $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ where α and $\beta > 0$ are real and $i^2 = -1$. Formally this case is similar to the case I:

$$y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$$

Since this is a solution for any choice of the constants C_1 and C_2, the choices $C_1 = C_2 = 1$ and $C_1 = 1$ and $C_2 = -1$ give two solutions

$$\begin{aligned} y_1 &= e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x} = e^{\alpha x}\left(e^{i\beta x} + e^{-i\beta x}\right) = 2e^{\alpha x}\cos\beta x \\ y_2 &= e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x} = e^{\alpha x}\left(e^{i\beta x} - e^{-i\beta x}\right) = 2ie^{\alpha x}\sin\beta x \end{aligned}$$

where we used the Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

The last two results show that $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are real solutions of (14) and form the fundamental set on $(-\infty, \infty)$. The general solution is

$$y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x).$$

Example 1: Solve the following DEs:

$$\begin{aligned} 2y'' - 5y' - 3y &= 0 \\ y'' - 10y' + 25y &= 0 \\ y'' + 4y' + 7y &= 0 \end{aligned}$$

Example 2: Solve the following IVP:

$$4y'' + 4y' + 17y = 0, \qquad y(0) = -1, \;\; y'(0) = 2$$

Example 3:

$$y'' + k^2 y = 0 \qquad y'' - k^2 y = 0$$

where k is real.

Undetermined coefficients

To solve a nonhomogeneous linear DE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = g(x)$$

we must

- find the complementary function y_c; and
- find *any* particular solution y_p of the nonhomogeneous equation.

The general solution on an interval I is $y = y_c + y_p$ where y_c is the solution of the associated homogeneous DE:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

Method of undetermined coefficients

To obtain a particular solution y_p we will make an educated guess about the form of y_p motivated by the kind of function that makes up the input function $g(x)$.

The general method is limited to nonhomogeneous linear DE where

- the coefficients, $a_i, i = 0, 1, ..., n$ are constants, and
- where $g(x)$ is a constant, a polynomial function, an exponential function $e^{\alpha x}$, sine or cosine functions $\sin\beta x$ or $\cos\beta x$, or finite sums and products of these functions.

The method of undetermined coefficients is not applicable to equations of the form (15) if

$$g(x) = \ln x, \qquad g(x) = \frac{1}{x}, \qquad g(x) = \tan x, \qquad g(x) = \sin^{-1} x$$

Example 1: Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6$.

Solution:

Step 1: Solve associated homogeneous equation $y'' + 4y' - 2y = 0$.
We find the roots of the auxiliary equatin $m^2 + 4m - 2 = 0$ are $m_1 = -2 - \sqrt{6}$ and $m_2 = -2 + \sqrt{6}$. The complementary function is thus

$$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}$$

$$y'' + 4y' - 2y = 2x^2 - 3x + 6$$

Step 2: Since $g(x)$ is quadratic polynomial, let us assume a particular solution in the form

$$y_p = Ax^2 + Bx + C$$

We wish to determine the coefficients A, B, and C for which y_p is a solution of the equation above:

$$y_p'' + 4y_p' - 2y_p = 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6$$

The coefficients of like powers of x must be equal, that is

$$-2A = 2, \quad 8A - 2B = -3, \quad 2A + 4B - 2C = 6$$

This leads to $A = -1$, $B = -5/2$, and $C = -9$, so this particular solution is

$$y_p = -x^2 - \frac{5}{2}x - 9. \tag{15}$$

Step 3: The general solution is then

$$y = y_c + y_p = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9. \tag{16}$$

Example 2: Particular solution using undetermined coefficients
Find a particular solution of $y'' - y' + y = 2\sin 3x$.

Example 3: Forming y_p by superposition
Find a particular solution of $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$.

A glitch in the method:

Example 4: Find a particular solution of $y'' - 5y' + 4y = 8e^x$.

Differentiation of e^x produces no new function, so proceeding with the particular solution assumed in the form of $y_p = Ae^x$ leads to a contradiction $0 = 8e^x$.

In fact our y_p is already contained in $y_c = c_1e^x + c_2e^{4x}$. Let us see whether we can find a particular solution of the form

$$y_p = Axe^x$$

Substituting this solution into the DE and simplifying gives

$$y_p'' - 5y_p' + 4y_p = -3Ae^x = 8e^x \quad \text{so} \quad y_p = -\frac{8}{3}xe^x$$

We distinguish two cases:

Case I: No function in the assumed particular solution is a solution of the associated homogeneous differential equation.

Case II: A function in the assumed particular solution is also a solution of the associated homogeneous differential equation.

Trial particular solutions

	$g(x)$	Form of y_p
1.	1 (any constant)	A
2.	$5x + 7$	$Ax + B$
3.	$3x^2 - 2$	$Ax^2 + Bx + C$
4.	$x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5.	$\sin 4x$	$A\cos 4x + B\sin 4x$
6.	$\cos 4x$	$A\cos 4x + B\sin 4x$
7.	e^{5x}	Ae^{5x}
8.	$(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9.	x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10.	$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11.	$5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
12.	$xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$

Example 5: Forms of particular solution - Case I

Determine the form of a particular solution of

$$\begin{aligned} y'' - 8y' + 25y &= 5x^3 e^{-x} - e^{-x} \\ y'' + 4y &= x \cos x \end{aligned}$$

If $g(x)$ consists of a sum of m terms of the kind listed in the table above, the assumption for a particular solution y_p consists of the sum of the trial forms $y_{p_1}, y_{p_2}, ..., y_{p_n}$ corresponding to these terms:

$$y_p = y_{p_1} + y_{p_2} + ... + y_{p_n}$$

The form rule for Case I: The form of y_p is a linear combination of all linearly independent functions that are generated by repeated differentiations of $g(x)$.

Example 6: Forming y_p by superposition - Case I

Determine the form of a particular solution of

$$y'' - 9y' + 14y = 3x^2 - 5\sin 2x + 7xe^{6x}$$

Solution:

$$\begin{aligned} 3x^2 &\Rightarrow y_{p_1} = Ax^2 + Bx + C \\ -5\sin 2x &\Rightarrow y_{p_2} = E\cos 2x + F\sin 2x \\ 7xe^{6x} &\Rightarrow y_{p_3} = (Gx + H)e^{6x} \end{aligned}$$

$$y = y_{p_1} + y_{p_2} + y_{p_3} = Ax^2 + Bx + C + E\cos 2x + F\sin 2x + (Gx + H)e^{6x}$$

No term in this solution duplicates a term in $y_c = c_1e^{2x} + c_2e^{7x}$.

Example 7: Particular solution - Case II

Find a particular solution of

$$y'' - 2y' + y = e^x$$

The complementary function is $y_c = c_1 e^x + c2xe^x$. Therefore we can not assume the particular solution in the form $y_p = Ae^x$ or $y_p = Axe^x$ since these would duplicate the terms in y_c. We try

$$y_p = Ax^2 e^x$$

Substituting this into the DE gives $2Ae^x = e^x$ and so $A = \frac{1}{2}$. The particular solution is $y_p = \frac{1}{2}x^2 e^x$.

Suppose again that $g(x)$ consists of m terms given by the table above, and that a particular solution y_p consists of the sum:

$$y_p = y_{p_1} + y_{p_2} + ... + y_{p_n}$$

where $y_{p_i}, i = 1, 2, ..., m$ are the corresponding trial solution forms.

Multiplication rule for Case II: If any y_{p_i} contains terms that duplicate terms in y_c then that y_{p_i} must be multiplied by x^n, where n is the smallest positive integer that eliminates that duplication.

Example 8: An IVP

$$y'' + y = 4x + 10 \sin x, \qquad y(\pi) = 0, \;\; y'(\pi) = 2 \tag{17}$$

The solution of the associated homogeneous equation $y'' + y = 0$ is $y_c = c_1 \cos x + c_2 \sin x$. To avoid duplication we use

$$y_p = Ax + B + Cx \cos x + Ex \sin x$$

The final solution of the IVP:

$$y = 9\pi \cos x + 7 \sin x + 4x - 5x \cos x$$

Example 9: Using the multiplication rule, solve

$$y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$$

The solution of the associated homogeneous equation is $y_c = c_1 e^{3x} + c_2 x e^{3x}$, so we choose the operative form of the particular solution to be

$$y_p = Ax^2 + Bx + C + Ex^2 e^{3x}$$

Substituting into the differential equation and collecting like terms gives $A = \frac{2}{3}$, $B = \frac{8}{9}$, $C = \frac{2}{3}$, and $E = -6$. The general solution is then

$$y = y_c + y_p = c_1 e^{3x} + c_2 x e^{3x} + \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3} - 6x^2 e^{3x}$$