First order ODEs II

- Linear first-order differential equations
- Method of variation of parameters
- Solutions by substitutions
- Bernoulli equation
- Reduction to separation of variables
- Optional material: error function, exact DE, homogeneous functions

Linear equations

A differential equation that is of the first degree in the dependent variable and all its derivatives is said to be linear.

Definition: Linear equation

A first-order differential equation of the form

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x) \tag{4}$$

is said to be linear.

If $g(x) = 0$ the linear equation is said to be **homogeneous**, otherwise it is **nonhomogeneous**.

Standard form

By dividing both sides of (4) by $a_1(x)$ we get the **standard form** of a linear equation

$$\frac{dy}{dx} + P(x)y = f(x) \tag{5}$$

We seek a solution of the equation above on an interval I for which both functions P and f are continuous.

The property

The DE (5) has the property that its solution is the **sum** of two solutions, $y = y_c + y_p$, where y_c is the solution of the associated homogeneous equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = 0 \tag{6}$$

and y_p is a particular solution of the nonhomogeneous equation (5). To see this

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y_c + y_p\right] + P(x)\left[y_c + y_p\right] = \left[\frac{\mathrm{d}y_c}{\mathrm{d}x} + P(x)y_c\right] + \left[\frac{\mathrm{d}y_p}{\mathrm{d}x} + P(x)y_p\right] = 0 + f(x) = f(x)$$

The homogeneous equation (6) is also separable, so we can find y_c by integrating it

$$y_c = ce^{-\int P(x)dx} = cy_1$$

We now use the fact that $dy_1/dx + P(x)y = 0$ to determine y_p.

The procedure: Variation of parameters

Idea: to find a function u so that $y_p = u(x)y_1(x) = u(x)e^{-\int P(x)dx}$ is a solution of (5).

Substituting $y_p = uy_1$ into the equation gives

$$
\begin{aligned}
u\frac{\mathrm{d}y_1}{\mathrm{d}x} + y_1\frac{\mathrm{d}u}{\mathrm{d}x} + P(x)uy_1 &= f(x) \\
u\left[\frac{\mathrm{d}y_1}{\mathrm{d}x} + P(x)y_1\right] + y_1\frac{\mathrm{d}u}{\mathrm{d}x} &= f(x)
\end{aligned}
$$

and since y_1 is the solution of the homogeneous equation, the expression in the square bracket is zero and

$$y_1\frac{\mathrm{d}u}{\mathrm{d}x} = f(x)$$

$$y_1 \frac{\mathrm{d}u}{\mathrm{d}x} = f(x)$$

Separating variables and integrating then gives

$$du = \frac{f(x)}{y_1(x)} dx \quad \Rightarrow \quad u = \int \frac{f(x)}{y_1(x)} dx$$

Since $y_1(x) = e^{-\int P(x)dx}$, $1/y_1(x) = e^{\int P(x)dx}$, and therefore

$$y_p = uy_1 = \left(\int \frac{f(x)}{y_1(x)} dx \right) e^{-\int P(x)dx} = e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x) dx$$

and the solution of (5) is then of the form

$$y = y_c + y_p = ce^{-\int P(x)dx} + e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)dx$$

There is an equivalent but easier way of solving (5). If the equation above is multiplied by $e^{\int P(x)dx}$ and differentiated we get

$$\begin{aligned} e^{\int P(x)dx} y &= c + \int e^{\int P(x)dx} f(x)dx \\ \frac{\mathrm{d}}{\mathrm{d}x}\left[e^{\int P(x)dx} y\right] &= e^{\int P(x)dx} f(x) \\ e^{\int P(x)dx} \frac{\mathrm{d}y}{\mathrm{d}x} + P(x)e^{\int P(x)dx} y &= e^{\int P(x)dx} f(x) \end{aligned}$$

Dividing the result by $e^{\int P(x)dx}$ gives (5).

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = f(x)$$

Method of solving a linear first-order equation

(i) Put a linear equation of form (4) into the standard form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = f(x)$$

and then determine $P(x)$ and the **integrating factor** $e^{\int P(x)dx}$.

(ii) Multiply the equation in its standard form by the integrating factor. The left side of the resulting equation is automatically the derivative of the integrating factor and y: write

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x)$$

and then integrate both sides of this equation.

Example:

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 6$$

The equation is already in the standard form. Since $P(x) = -3$, the integrating factor is $e^{\int(-3)dx} = e^{-3x}$. By multiplying the equation by the integrating factor, we get

$$e^{-3x}\frac{\mathrm{d}y}{\mathrm{d}x} - 3e^{-3x}y = 6e^{-3x}$$

which is the same as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-3x}y\right] = 6e^{-3x}$$

Integrating both sides of the equation yields $e^{-3x}y = -2e^{-3x} + c$, so the solution is

$$y = -2 + ce^{3x}$$

for $-\infty < x < \infty$. Note that the DE above is autonomous (a_0, a_1 and g are constants) and it has one unstable critical point at $y = -2$.

Constant of integration

Considering the constant of integration in evaluation of the integrating factor $e^{\int P(x)dx}$, that is writting $e^{\int P(x)dx + c}$ is unnecessary as the integrating factor multiplies both sides of the differential equation.

Singular points

The recasting the linear equation (4) in the standard form (5) requires division by $a_1(x)$. Values of x for which the $a_1(x) = 0$ are called **singular points**. They are potentially troublesome: if $P(x)$ formed by dividing $a_0(x)$ by $a_1(x)$ is discontinuous at a point, the discontinuity may carry over to solutions of the DE.

General solution

Recall that the functions $P(x)$ and $f(x)$ in (5) are continuous on a common interval I. Also, *if* (5) has a solution on I it must be of the form

$$y = y_c + y_p = ce^{-\int P(x)dx} + e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)dx$$

Conversely any function of this form is a solution of (5) on I. In other words, the solution above defines a one-parameter family of solutions of equation (5) and every solution of (5) defined on I is of this form. It is hence the **general solution**.

Now writing (5) in the normal form $y' = F(x, y)$, we see that $F(x, y) = -P(x)y + f(x)$ and $\partial F/\partial y = -P(x)$. These must be continuous on the entire interval I because of the continuity of $P(x)$ and $f(x)$.

From the uniqueness theorem we conclude that there exists one and only one solution of the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = f(x), \qquad y(x_0) = y_0$$

defined on some interval I_0 containing x_0 and that this interval of existence and uniqueness is the entire interval I.

Example: General solution

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = x^6 e^x$$

The standard from

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{4}{x}y = x^5 e^x$$

from which $P(x) = -4/x$, $f(x) = x^5 e^x$ and both are continuous on $(0, \infty)$. The integrating factor is then

$$e^{-4\int dx/x} = e^{-4\ln x} = e^{lnx^{-4}} = x^{-4}$$

We multiply the standard form by x^{-4} and integrate by parts

$$x^{-4}\frac{\mathrm{d}y}{\mathrm{d}x} - 4x^{-5}y = xe^x \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-4}y\right] = xe^x \quad \Rightarrow \quad x^{-4}y = xe^x - e^x + c$$

The general solution defined on $(0, \infty)$ is then $y = x^5 e^x - x^4 e^x + cx^4$.

Example: General solution

$$\begin{aligned}(x^2-9)\frac{\mathrm{d}y}{\mathrm{d}x}+xy &= 0\\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}+\frac{x}{x^2-9}y &= 0\end{aligned}$$

Thus $P(x) = x/(x^2-9)$. Although it is continuous on $(-\infty,-3)$, $(-3,3)$, and $(3,\infty)$, we will solve it at the first and the third interval on which the integrating factor is

$$e^{\int x dx/(x^2-9)} = e^{1/2\int 2x dx/(x^2-9)} = e^{1/2\ln|x^2-9|} = \sqrt{x^2-9}$$

After multiplying the standard form by the integrating factor and integrating we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{x^2-9}y\right] = 0 \quad\Rightarrow\quad \sqrt{x^2-9}y = c$$

thus for either $x < -3$ or $x > 3$, the general solution is $y = c/\sqrt{x^2-9}$.

Example: An IVP

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = x, \qquad y(0) = 4$$

$P(x) = 1$, $f(x) = x$ are continuous on $(-\infty, \infty)$. The integrating factor is $e^{\int dx} = e^x$, and so integrating

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^x y\right] = xe^x$$

gives $e^x y = xe^x - e^x + c$. The general solution is then $y = x - 1 + ce^{-x}$. From the initial condition, $y(0) = 4$ we obtain the value of the integrating constant $c = 5$, and thus the solution of our IVP is

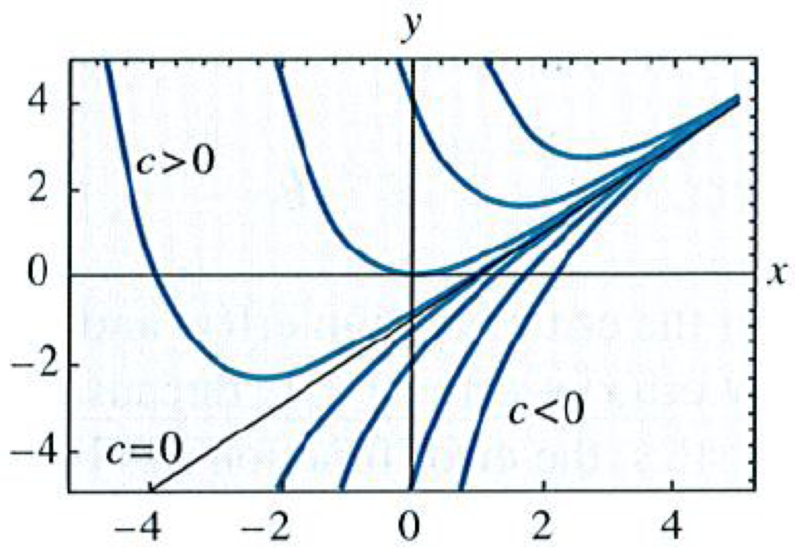
$$y = x - 1 + 5e^{-x}, \qquad -\infty < x < \infty$$

The general solution of every linear first order DE is a sum, $y = y_c + y_p$, of the solution of the associated homogeneous equation (6) and a particular solution of the nonhomogeneous equation.

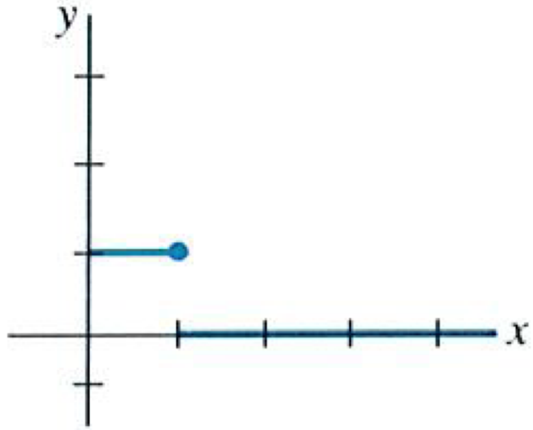
In the example above, $y_c = ce^{-x}$ and $y_p = x - 1$.

Observe that as x gets large, the graphs of all members of the family get close to the graph of y_p, as y_c becomes negligible.

We say $y_c = ce^{-x}$ is a **transient term** since $y_c \to 0$ as $x \to \infty$.



Example: A discontinuous $f(x)$



$$\frac{dy}{dx} + y = f(x)$$

where $f(x) = 1$ for $0 \le x \le 1$, and $f(x) = 0$ for $x > 1$; the initial condition is $y(0) = 0$.

We solve the problem in two intervals over which f is defined. For $0 \le x \le 1$:

$$\frac{dy}{dx} + y = 1 \quad \Rightarrow \quad \frac{d}{dx}[e^x y] = e^x$$

we get $y = 1 + c_1 e^{-x}$ and since $y(0) = 0$ we have $c_1 = -1$, and so $y = 1 - e^{-x}$.

For $x > 1$ the equation

$$\frac{dy}{dx} + y = 0$$

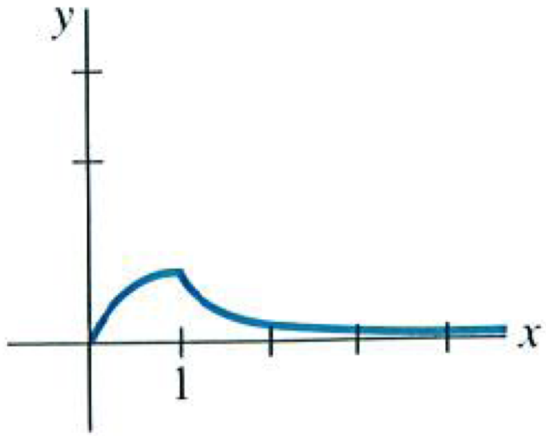
leads to the solution $y = c_2 e^{-x}$. So the solution in both intervals is

$$y = \begin{cases} 1 - e^{-x} & \text{if } 0 \le x \le 1; \\ c_2 e^{-x} & \text{if } x > 1. \end{cases}$$

In order for y to be continuous, we want $\lim_{x \to 1^+} y(x) = y(1)$, that is, $c_2 e^{-1} = 1 - e^{-1}$ or $c_2 = e - 1$. The function

$$y = \begin{cases} 1 - e^{x} & \text{if } 0 \le x \le 1; \\ (e - 1)e^{-x} & \text{if } x > 1. \end{cases}$$

is continuous on $(0, \infty)$.



Solutions by substitutions

We first transform a given differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

by means of **substitution** $y = g(x, u)$ into another differential equation

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= g_x(x, u) + g_u(x, u)\frac{\mathrm{d}u}{\mathrm{d}x} \\ f(x, g(x, u)) &= g_x(x, u) + g_u(x, u)\frac{\mathrm{d}u}{\mathrm{d}x} \end{aligned}$$

where we assumed that $g(x, u)$ possesses the first partial derivatives, so we could apply the chain rule.

The last equation above can be reformulated as $du/dx = F(x, u)$. If we can find its solution $u = \phi(x)$, then a solution of the original equation is $y = g(x, \phi(x))$.

Bernoulli equation

is a special type of first-order ODE which can be reduced to linear form and then solved by the method for linear ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + M(x)y = N(x)y^n \tag{7}$$

where n is any real number.

It can be transformed into the linear form as follows:

$$u = y^{1-n} \quad \Rightarrow \quad u = yy^{-n} \quad \Rightarrow \quad y = y^n u$$

Differentiating this gives

$$\frac{\mathrm{d}u}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} + M(x)y = N(x)y^n$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{y^n}{1-n}\right)\frac{\mathrm{d}u}{\mathrm{d}x}$$

Substituting this into the Bernoulli equation (7) gives

$$\left(\frac{y^n}{1-n}\right)\frac{\mathrm{d}u}{\mathrm{d}x} + M(x)\, y^n\, u = N(x)\, y^n$$

Dividing by y^n and multiplying by $(1-n)$ gives

$$\frac{\mathrm{d}u}{\mathrm{d}x} + (1-n)M(x)u = (1-n)N(x)$$

which is a linear ODE with $P(x) = (1-n)M(x)$ and $f(x) = (1-n)N(x)$.

$$\frac{dy}{dx} + M(x)y = N(x)y^n$$

Example: A Bernoulli equation

$$\frac{dy}{dx} + \frac{1}{3}y = \frac{1}{3}(1 - 2x)y^4$$

where $n = 4$, $M(x) = 1/3$, and $N(x) = (1 - 2x)/3$. Using the transformation

$$u = y^{1-n} = y^{-3}$$

we obtain the linear ODE

$$\begin{aligned} \frac{du}{dx} + (1-n)M(x)u &= (1-n)N(x) \\ \Rightarrow \quad \frac{du}{dx} - u &= (2x - 1) \end{aligned}$$

whose solution is $u(x) = ce^x - (2x + 1)$; the solution of the original equation is then $y = 1/\left[ce^x - (2x + 1)\right]^{1/3}$.

Reduction to separation of variables

A differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(Ax + By + C)$$

can always be reduced to an equation with separable variables by means of the substituition $u = Ax + By + C$.

Example: An IVP

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (-2x + y)^2 - 7, \qquad y(0) = 0$$

Let $u = -2x + y$, then $du/dx = -2 + dy/dx$ and so the DE is transformed into a separable equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} + 2 = u^2 - 7 \quad \text{or} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = u^2 - 9$$

The transformed equation can be solved using the partial fractions

$$\frac{du}{(u-3)(u+3)} = dx \qquad \text{or} \qquad \frac{1}{6}\left[\frac{1}{u-3} - \frac{1}{u+3}\right]du = dx$$

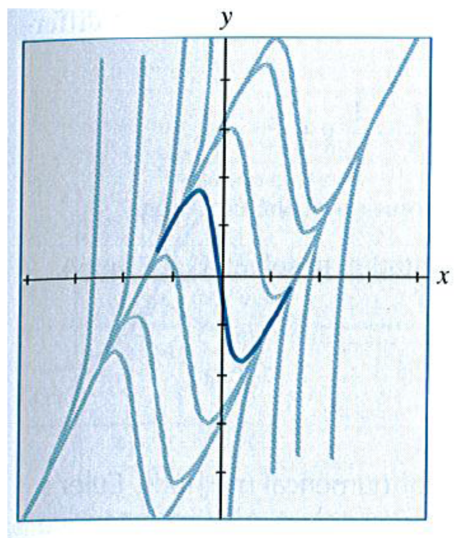
$$\Rightarrow \qquad \frac{1}{6}\ln\left|\frac{u-3}{u+3}\right| = x + c_1 \qquad \text{or} \qquad \frac{u-3}{u+3} = e^{6x+6c_1}$$

After solving the last equation for u and then resubstituting we get

$$u = \frac{3(1+ce^{6x})}{1-ce^{6x}} \qquad \text{or} \qquad y = 2x + \frac{3(1+ce^{6x})}{1-ce^{6x}}$$

and by applying the initial condition we get $c = -1$

$$y = 2x + \frac{3(1-e^{6x})}{1+e^{6x}}$$



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Functions defined by integrals

Integrals of functions, which do not possess indefinite integrals that are elementary functions, are called **nonelementary**.

Error function

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Complementary error function

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

Since $2/\sqrt{\pi} \int_0^\infty e^{-t^2} dt = 1$, $erf(x) + erfc(x) = 1$. Also $erf(0) = 0$.

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Example: The error function

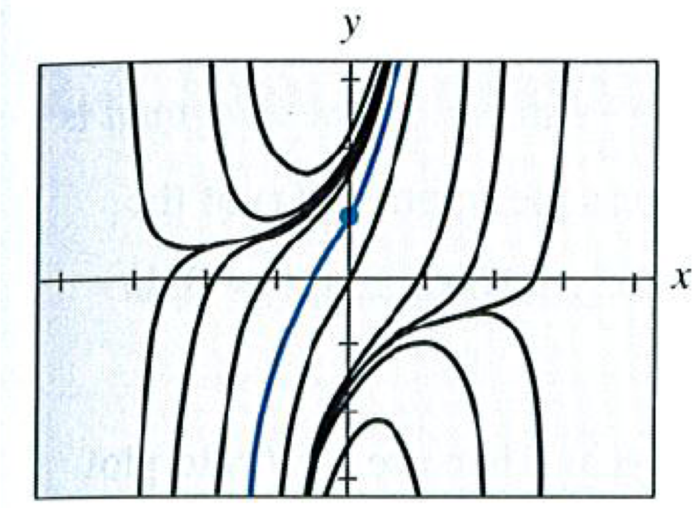
$$\frac{dy}{dx} - 2xy = 2, \qquad y(0) = 1$$

The integrating factor is e^{-x^2}, and so from

$$\frac{d}{dx}\left[e^{-x^2}y\right] = 2e^{-x^2} \qquad \Rightarrow \qquad y = 2e^{x^2}\int_0^x e^{-t^2}dt + ce^{x^2}$$

From the initial value we get $c = 1$ and thus the solution of the IVP is

$$y = 2e^{x^2}\int_0^x e^{-t^2}dt + e^{x^2} = e^{x^2}\left[1 + \sqrt{\pi}\, erf(x)\right]$$



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Exact equations

A differential expression $M(x, y)dx + N(x, y)dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$, i.e.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

A first order differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be an **exact equation** if the expression on the l. h. s. is an exact differential.

Example: $x^2y^3dx + x^3y^2dy = 0$ is exact as $d(x^3y^3/3) = x^2y^3dx + x^3y^2dy$.

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Theorem: Criterion for an exact differential

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in the region R defined by $a < x < b$ and $c < y < d$. Then a necessary and sufficient condition that $M(x, y)dx + N(x, y)dy$ be an exact differential is

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

<u>Proof:</u>

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial N}{\partial x}$$

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Example: Solution of an exact equation

$$2xydx + (x^2 - 1)dy = 0$$

$M(x, y) = 2xy$ and $N(x, y) = x^2 - 1$, we get $\partial M/\partial y = 2x = \partial N/\partial x$, so the equation is exact and there exist a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 2xy \quad \text{or} \quad \frac{\partial f}{\partial y} = x^2 - 1$$

Integrating the first equation gives

$$f(x, y) = x^2y + g(y)$$

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By taking now the partial derivative w.r.t. y we obtain

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1$$

from which it follows that $g'(y) = -1$ and $g(y) = -y$.

Hence $f(x, y) = x^2y - y$, and so the solution of the equation in implicit form is

$$x^2y - y = c$$

The explicit solution is $y = c/(x^2 - 1)$ and is defined on any interval not containing $x = \pm 1$.

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Homogeneous equations

A first-order DE in differential form

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be **homogeneous** if both coefficients M and N are **homogeneous functions** of the same degree α, i.e.

$$M(tx, ty) = t^{\alpha} M(x, y) \qquad N(tx, ty) = t^{\alpha} N(x, y)$$

Introducing $u = y/x$ and $v = x/y$, we can rewrite the coefficients as

$$M(x, y) = x^{\alpha} M(1, u) \qquad N(x, y) = x^{\alpha} N(1, u)$$
$$M(x, y) = y^{\alpha} M(v, 1) \qquad N(x, y) = y^{\alpha} N(v, 1)$$

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Either of the substitutions above, $y = ux$ or $x = vy$, will reduce a homogeneous equation to a *separable* first order ODE:

$$\begin{aligned} M(x,y)dx + N(x,y)dy &= 0 \\ \Rightarrow \quad x^\alpha M(1,u)dx + x^\alpha N(1,u)dy &= 0 \\ \Rightarrow \quad M(1,u)dx + N(1,u)dy &= 0 \end{aligned}$$

By substituting the differential $dy = udx + xdu$, we get a separable DE in the variables u and x:

$$\begin{aligned} M(1,u)dx + N(1,u)\,[udx + xdu] &= 0 \\ [M(1,u) + uN(1,u)]\,dx + xN(1,u)du &= 0 \\ \Rightarrow \quad \frac{dx}{x} + \frac{N(1,u)\,du}{M(1,u) + u\,N(1,u)} &= 0 \end{aligned}$$

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Example: Solving a homogeneous DE

$$\left(x^2 + y^2\right) dx + \left(x^2 - xy\right) dy = 0$$

The coefficients $M(x, y) = x^2 + y^2$ and $N(x, y) = x^2 - xy$ are homogeneous functions of the degree 2. Let $y = ux$, then $dy = udx + xdu$, and the given DE becomes

$$\begin{aligned} \left(x^2 + u^2x^2\right) dx + \left(x^2 - ux^2\right) [udx + xdu] &= 0 \\ x^2 (1 + u)\, dx + x^3 (1 - u)\, du &= 0 \\ \frac{1 - u}{1 + u} du + \frac{dx}{x} = 0 \\ \left[-1 + \frac{2}{1 + u}\right] du + \frac{dx}{x} = 0 \end{aligned}$$

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$$\left[-1 + \frac{2}{1+u}\right] du + \frac{dx}{x} = 0$$

After integration, and transformation back to the original variables, we get

$$-u + 2\ln|1+u| + \ln|x| = \ln|c| \qquad \Rightarrow \qquad -\frac{y}{x} + 2\ln\left|1 + \frac{y}{x}\right| + \ln|x| = \ln|c|$$

Using the properties of logarithms, the solution can be written as $(x + y)^2 = cxe^{y/x}$.

Intuitive interpretation of a linear ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = f(x)$$

The function $f(x)$ often represents some controllable quantity, such as a force or an applied voltage, which can be interpreted as the **input** to the system. Within this interpretation, we can view the dependent variable $y(x)$ as an **output** or as an effect which is produced **in response** to the **input(s)**.

In the general solution of the linear ODE

$$y = e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)dx + ce^{-\int P(x)dx}$$

the first term can be viewed as the system response to the input $f(x)$ and the second term as the influence of the initial state of the system.

Modelling an RC-circuit

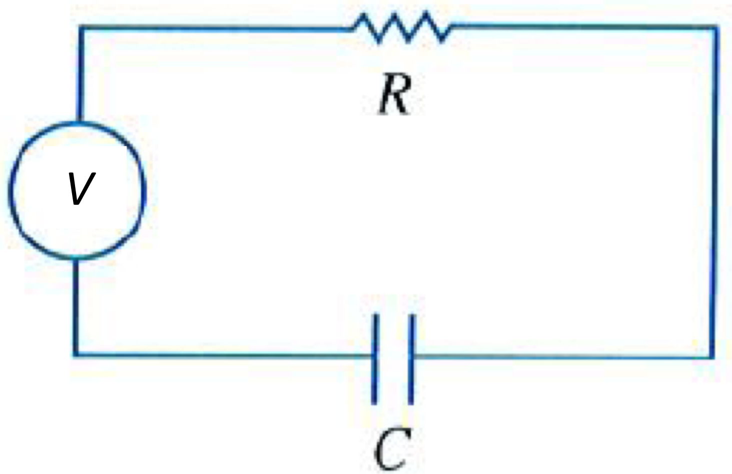
A resistor of resistance R is connected in series with a capacitor of capacitance C and a source of electromotive force in the form of an applied voltage, $V(t)$. When the circuit is closed, a current $i(t)$ will flow through it.

According to the Kirchhoff second law with this circuit, the voltage drops at the capacitor and resistor equal the applied voltage:

$$V_R + V_C = V(t)$$

where $V_R = Ri$ and $V_C = q/C = \int idt/C$. Thus we get

$$Ri + \frac{1}{C}\int idt = V(t)$$



Let us differentiate w.r.t. t and divide by R, to get

$$\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{RC}i = \frac{1}{R}\frac{\mathrm{d}V(t)}{\mathrm{d}t}$$

This equation has the form which is the standard form of the linear equation where $P(t) = 1/RC$ and $f(t) = (1/R)dV(t)/dt$. The integrating factor is then

$$e^{\int \frac{1}{RC}dt} = e^{\frac{t}{RC}}$$

so the general solution becomes

$$i(t) = e^{-\frac{t}{RC}}\left(\frac{1}{R}\int e^{\frac{t}{RC}}\frac{\mathrm{d}V(t)}{\mathrm{d}t}dt + c\right)$$

Case 1: $V(t) =$ **constant**

In this case we get $\frac{\mathrm{d}V(t)}{\mathrm{d}t} = 0$ and so

$$i(t) = ce^{-\frac{t}{RC}}$$

The current in this case decays with time eventually approaching zero

Case 2: $V(t) = V_0 \sin(\omega t)$
Substituting this into the general form of the solution we get

$$i(t) = e^{-\frac{t}{RC}} \left(\frac{1}{R} \int e^{\frac{t}{RC}} V_0 \omega \cos(\omega t) dt + c \right)$$

Integrating by parts and using trigonometric relations gives

$$\begin{aligned} i(t) &= ce^{-\frac{t}{RC}} + \frac{\omega V_0 C}{1 + (\omega RC)^2} [\cos(\omega t) + \omega RC \sin(\omega t)] \\ &= ce^{-\frac{t}{RC}} - \frac{\omega V_0 C}{\sqrt{1 + (\omega RC)^2}} \sin(\omega t - \phi) \end{aligned}$$

where $\tan(\phi) = -1/\omega RC$.
The response involves two terms: an exponential decay and steady state response to oscillating external voltage, oscillating with ω and the amplitude $\frac{\omega V_0 C}{\sqrt{1+(\omega RC)^2}}$.