Introduction to differential equations: overview

- Definition of differential equations and their classification
- Solutions of differential equations
- Initial value problems
- Existence and uniqueness
- Mathematical models and examples
- Methods of solution of first-order differential equations

Definition: Differential Equation

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation**.

Examples:

$$(i)\ \frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + y^2 = 0 \qquad (ii)\ y'' - 2y' + y = 0 \qquad (iii)\ \ddot{s} = -32 \qquad (iv)\ \frac{\partial^2 u}{\partial x^2} = -2\frac{\partial u}{\partial t}$$

Classification of differential equations

(a) Classification by Type:

Ordinary differential equations - ODE

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0$$

Partial differential equations - PDE

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$$

(b) Classification by Order:

The **order** of the differential equation is the order of the highest derivative in the equation.

Example:

nth-order ODE:

$$F\left(x, y, y', ..., y^{(n)}\right) = 0 \tag{1}$$

Normal form of (1)

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = f\left(x, y, y' ..., y^{(n-1)}\right)$$

(c) Classification as Linear or Non-linear:

An nth-order ODE (1) is said to be **linear** if it can be written in this form

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + ... + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

Examples:

Linear: $(y - x)dx + 4xdy = 0$ $\quad y'' - 2y' + y = 0$ $\quad \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 3x\frac{\mathrm{d}y}{\mathrm{d}x} - 5y = e^x$

Nonlinear: $\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + y^2 = 0$ $\quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \sin(y) = 0$ $\quad (1 - y)y' + 2y = e^x$

Solution of an ODE:

Any function ϕ defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an n-th-order ordinary differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

In other words:
a solution of an nth-order ODE is a function ϕ that possesses at least n derivatives and

$$F\left(x, \phi(x), \phi'(x), ..., \phi^{(n)}(x)\right) = 0 \tag{2}$$

for all $x \in I$. Alternatively we can denote the solution as $y(x)$.

Interval of definition:

A *solution* of an ODE has to be considered simultaneously with the *interval* I which we call

the interval of definition
the interval of existence,
the interval of validity, or
the domain of the solution.

It can be an open interval (a, b), a closed interval $[a, b]$, an infinite interval (a, ∞) and so on.

Example:

Verify that the function $y = xe^x$ is a solution of the differential equation $y'' - 2y' + y = 0$ on the interval $(-\infty, \infty)$:

From the derivatives

$$\begin{aligned} y' &= xe^x + e^x \\ y'' &= xe^x + 2e^x \end{aligned}$$

we see

$$\begin{aligned} &l.h.s.: \quad && y'' - 2y' + y = (xe^x + 2e^x) - 2\,(xe^x + e^x) + xe^x = 0 \\ &r.h.s.: \quad && 0 \end{aligned}$$

that each side of the equation is the same for every real number x.

A solution that is identically zero on an interval I, i.e. $y = 0, \forall x \in I$, is said to be **trivial**.

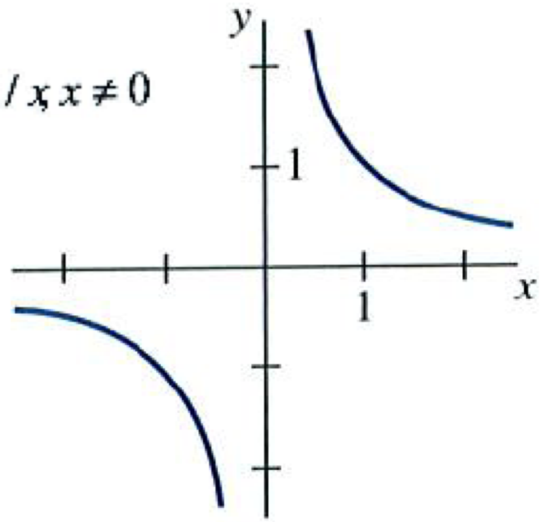
Solution curve:

is the graph of a solution ϕ of an ODE.

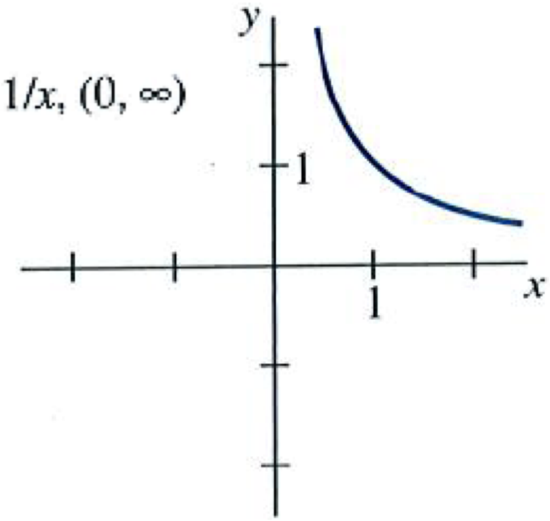
The graph of the solution ϕ is NOT the same as the graph of the functions ϕ as the domain of the function ϕ does not need to be the same as the interval I of definition (domain) of the solution ϕ.

Example:

(a) Function $y = 1/x$, $x \neq 0$



(b) Solution $y = 1/x$, $(0, \infty)$



Explicit solutions:

a solution in which the dependent variable is expressed solely in terms of the independent variable and constants.

Example:

$$y = \phi(x) = e^{0.1\ x^2}$$

is an explicit solution of the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0.2xy$$

Implicit solutions:

A relation $G(x, y) = 0$ is said to be an **implicit solution** of an ODE on an interval I provided there exists at least one function ϕ that satisfies the relation as well as the differential equation on I.

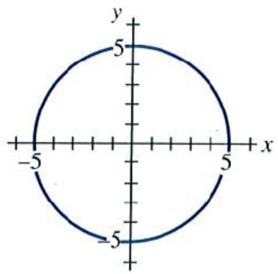
Example:

$$x^2 + y^2 = 25$$

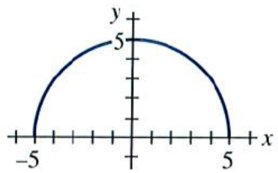
is an implicit solution of the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$$

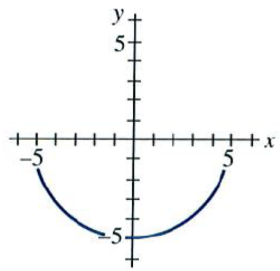
on the interval $(-5, 5)$.
Notice that also $x^2 + y^2 - c = 0$ satisfies the ODE above.



(a) Implicit solution
$x^2 + y^2 = 25$



(b) Explicit solution
$y_1 = \sqrt{25 - x^2}, -5 < x < 5$



(c) Explicit solution
$y_2 = -\sqrt{25 - x^2}, -5 < x < 5$

Families of solutions:

A solution ϕ of a first-order ODE $F(x, y, y') = 0$ can be referred to as an **integral** of the equation, and its graph is called an **integral curve**.

A solution containing an arbitrary constant (an integration constant) c represents a set

$$G(x, y, c) = 0$$

called a **one-parameter family of solutions**.

When solving an nth-order ODE $F(x, y, y', ..., y^{(n)}) = 0$, we seek an n-**parameter family of solutions** $G(x, y, c_1, c_2, ..., c_n) = 0$.

A single ODE can possess an infinite number of solutions!

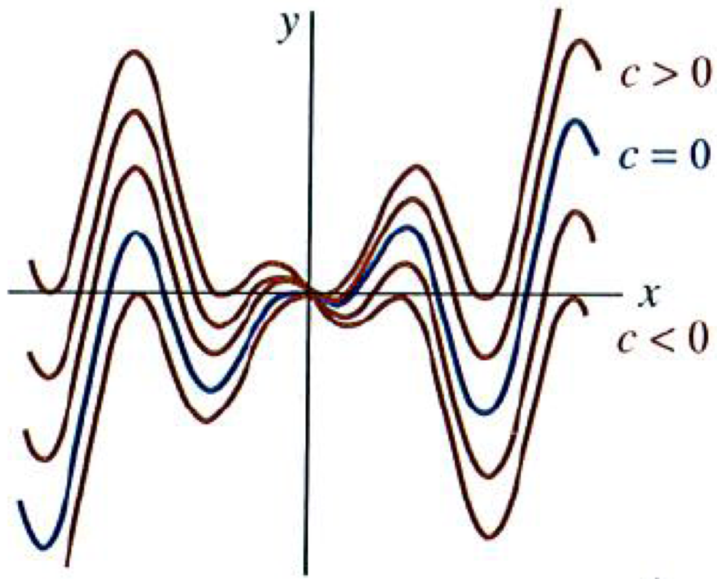
A particular solution:

is a solution of an ODE that is free of arbitrary parameters.

Example:

$y = cx - x\cos x$ is an explicit solution of $xy' - y = x^2 \sin x$ on $(-\infty, \infty)$.

The solution $y = -x\cos x$ is a particular solution corresponding to $c = 0$.



A singular solution:

a solution that can not be obtained by specializing any of the parameters in the family of solutions.

Example:

$y = (x^2/4 + c)^2$ is a one-parameter family of solutions of the DOE $dy/dx = xy^{1/2}$.

Also $y = 0$ is a solution of this ODE but it is not a member of the family above. It is a singular solution.

The general solution:

If every solution of an nth-order ODE $F(x, y, y', ..., y^{(n)}) = 0$ on an interval I can be obtained from an n-parameter family $G(x, y, c_1, c_2, ..., c_n) = 0$ by appropriate choices of the parameters c_i, $i = 1, 2, ..., n$ we then say that the family is the **general solution** of the differential equation.

Systems of differential equations:

A **system of ordinary differential equations** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable.

Example:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x, y)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(t, x, y)$$

A **solution** of a system, such as above, is a pair of differentiable functions $x = \phi_1(t)$ and $y = \phi_2(t)$ defined on a common interval I that satisfy each equation of the system on this interval.

Initial value problem:

On some interval I containing x_0, the problem of solving

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = f\left(x, y, y', ..., y^{(n)}\right)$$

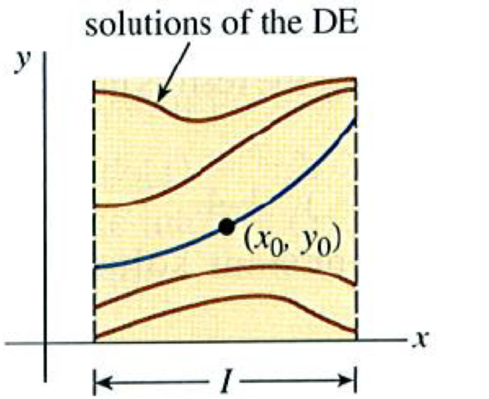
subject to the conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad ..., \quad y^{(n-1)}(x_0) = y_{n-1}$$

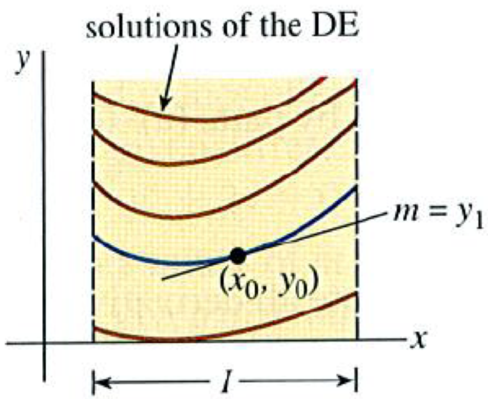
where y_0, y_1, ... , y_{n-1} are arbitrarily specified constants, is called an **initial value problem (IVP)**.

The conditions $y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(n-1)}(x_0) = y_{n-1}$ are called **initial conditions**.

First-order and Second-order IVPs:



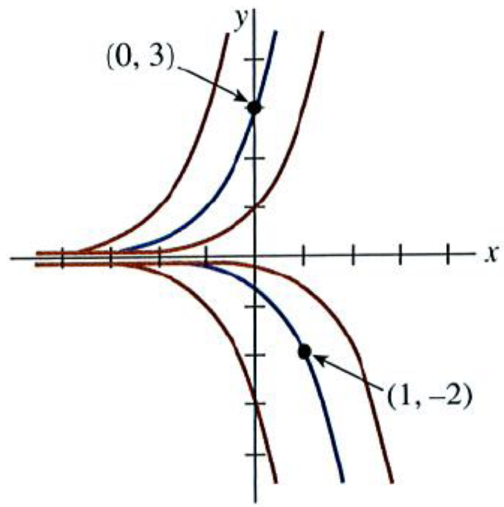
$$\frac{dy}{dx} = f(x, y)$$
$$y(x_0) = y_0 \qquad (3)$$



$$\frac{d^2y}{dx^2} = f(x, y, y')$$
$$y(x_0) = y_0$$
$$y'(x_0) = y_1 \qquad (4)$$

Example:

$y = ce^x$ is a one-parameter family of solutions of the first order ODE $y' = y$ on the interval $(-\infty, \infty)$.



The initial condition $y(0) = 3$ determines the constant c:

$$y(0) = 3 = ce^0 = c$$

Thus the function $y = 3e^x$ is a solution of the IVP defined by

$$y' = y, \quad y(0) = 3$$

Similarly, the initial condition $y(1) = -2$ will yield $c = -2e^{-1}$. The function $y = -2e^{x-1}$ is a solution of the IVP

$$y' = y, \quad y(1) = -2$$

Existence and uniqueness:

Does a solution of the problem exist? If a solution exist, is it unique?

Existence (for the IVP (3)):
Does the differential equation $dy/dx = f(x, y)$ *possess solutions?*
Do any of the solution curves pass through the point (x_0, y_0)*?*

Uniqueness (for the IVP (3)):
When can we be certain that there is precisely one solution curve passing through the point (x_0, y_0)*?*

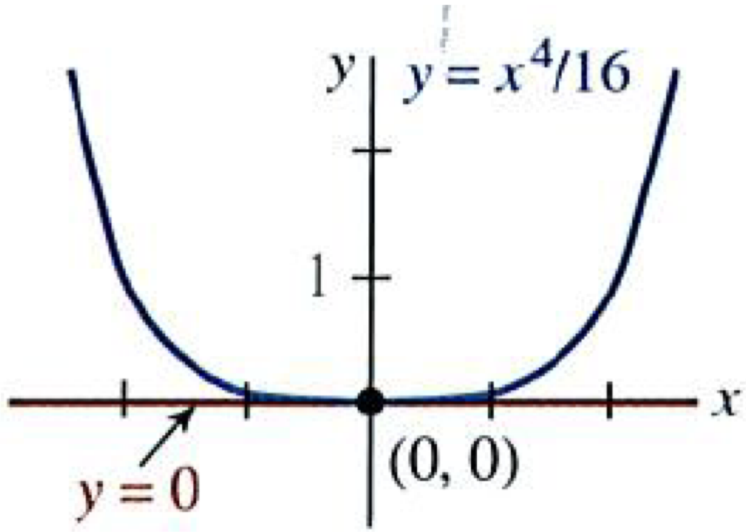
Example: An IVP can have several solutions

Each of the functions

$$\begin{aligned} y &= 0 \\ y &= x^4/16 \end{aligned}$$

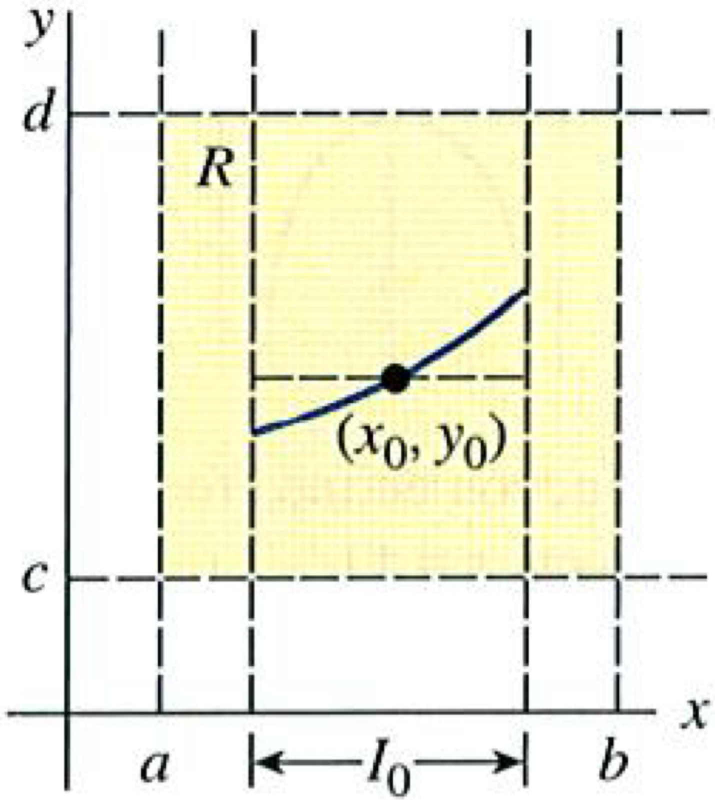
satisfy the IVP

$$\begin{aligned} \frac{dy}{dx} &= xy^{1/2} \\ y(0) &= 0 \end{aligned}$$



Theorem: Existence of a unique solution

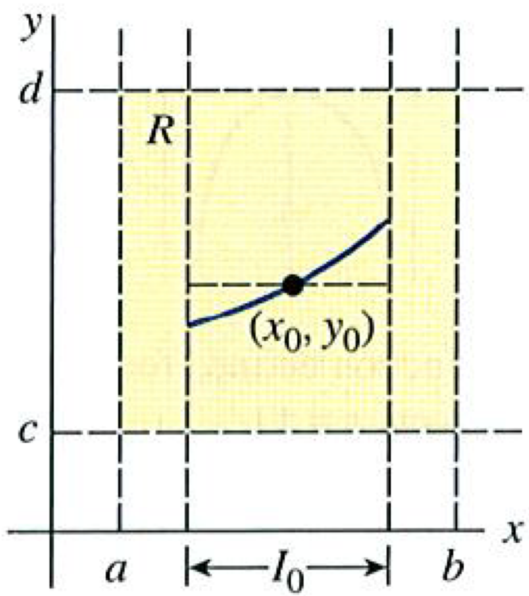
Let R be a rectangular region in the xy-plane defined by $a \leq x \leq b$, $c \leq y \leq d$, that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\partial f / \partial y$ are continuous on R, then there exist some interval I_0: $x_0 - h < x < x_0 + h$, $h > 0$, contained in $a \leq x \leq b$, and a unique function $y(x)$ defined on I_0, that is a solution of the initial value problem (3).



Distinguish the following three sets on the real x-axis:

the domain of the function $y(x)$;
the interval I over which the solution $y(x)$ is defined or exists;
the interval I_0 of existence AND uniqueness.

The theorem above gives no indication of the sizes of the intervals I and I_0; the number $h > 0$ that defines I_0 could be very small. Thus we should think that the solution $y(x)$ is *unique in a local sense*, that is near the point (x_0, y_0).



Example: uniqueness

Consider again the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy^{1/2}$$

in the light of the theorem above. The functions

$$f(x, y) = xy^{1/2}$$
$$\frac{\partial f}{\partial y} = \frac{x}{2y^{1/2}}$$

are continuous in the upper half-plane defined by $y > 0$.

The theorem allow us to conclude that through any point (x_0, y_0), $y_0 > 0$, in the upper half-plane, there is an interval centered at x_0, on which the ODE has a unique solution.

First-order differential equations

To find either explicit or implicit solution, we need to

(i) recognize the *kind* of differential equation, and then

(ii) apply to it an equation-specific method of solution.

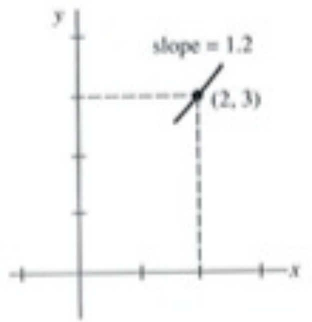
Solution curves without the solution

What a first order ODE can tell us?

Slope

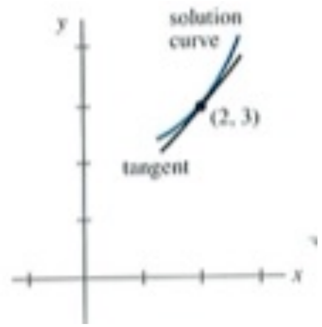
$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

The value $f(x, y)$ at the point (x, y) represents the slope of a **lineal element**, a miniature tangent line to the solution at that point.



Example:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0.2xy \\ f(x, y) &= 0.2xy \end{aligned}$$



At the point $(2, 3)$ the slope of a lineal element is $f(2, 3) = 1.2$.

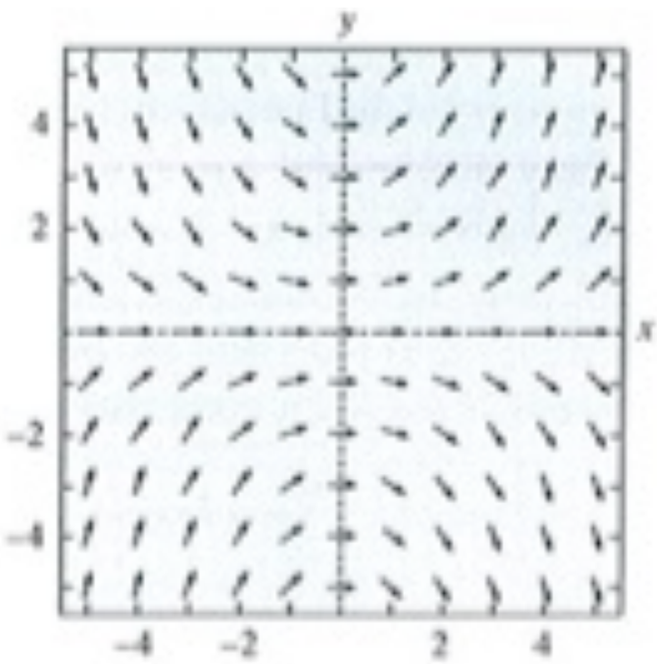
Direction fields or slope fields

is the collection of all lineal elements evaluated at each point (x, y) of a rectangular grid.

It provides the appearance or shape of a family of solution curves of the ODE and allows us to investigate its qualitative aspects.

<u>Example:</u>

$$\frac{dy}{dx} = 0.2xy$$



Increasing or decreasing solution

Increasing $y(x)$ if for all $x \in I$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} > 0$$

Decreasing $y(x)$ if for all $x \in I$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} < 0$$

Autonomous first-order DE

is DE in which the independent variable does not appear explicitly:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(y)$$

Examples:

Autonomous

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2$$

Non-autonomous

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0.2xy$$

Critical points

A real number c is a **critical point** of the autonomous DE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(y) \tag{1}$$

if it is a zero of f, i.e. $f(c) = 0$.

A critical point is also called an **equilibrium point** or **stationary point**.

If c is a critical point of (1), then $y(x) = c$ is a constant solution of the autonomous equation.

A constant solution $y(x) = c$ of (1) is called an **equilibrium solution**;
equilibria are the *only* constant solutions of (1).

Example: Autonomous ODE

$$\frac{dP}{dt} = P(a - bP)$$

where $a > 0$, $b > 0$. From $f(P) = P(a - bP) = 0$ we see that 0 and a/b are critical points of the equation.

By putting the critical points on a vertical line we obtain a **one-dimensional phase portrait** of the DE above.

We get three intervals:
$-\infty < P < 0, 0 < P < a/b, a/b < P < \infty$;
the arrows indicate the algebraic sign of $f(P) = P(a - bP)$
and whether a non-constant solution is increasing or decreasing.

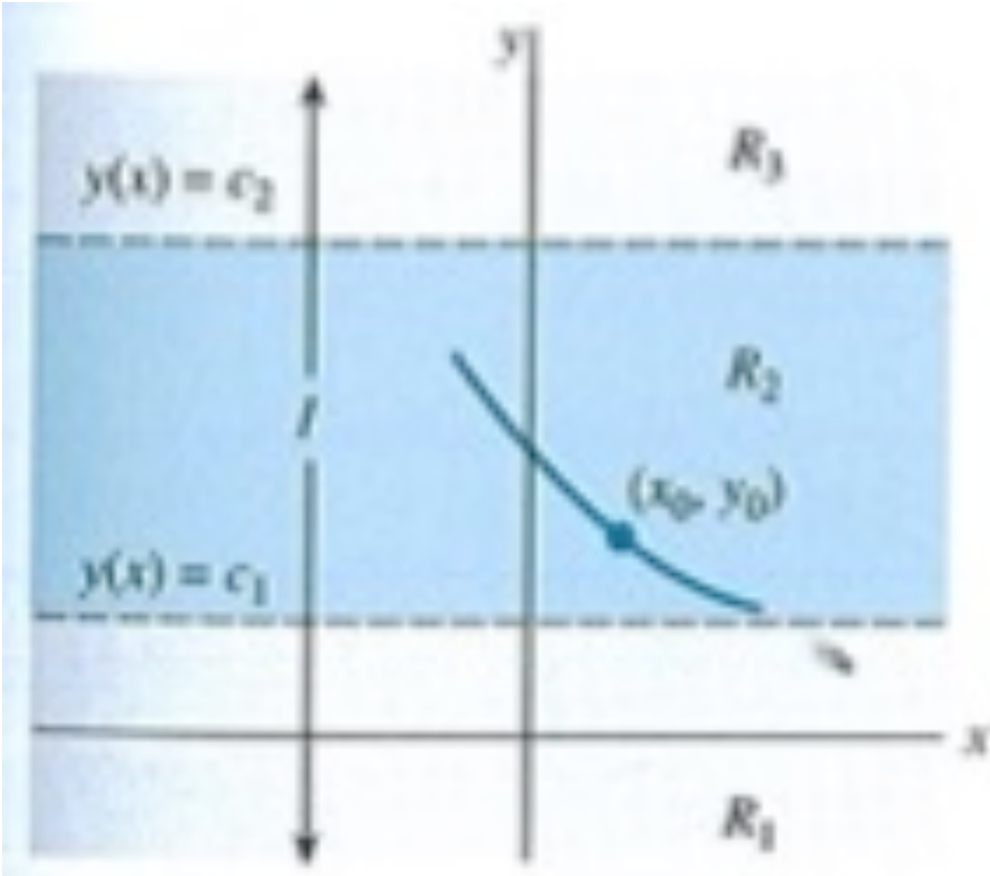
Solution curves

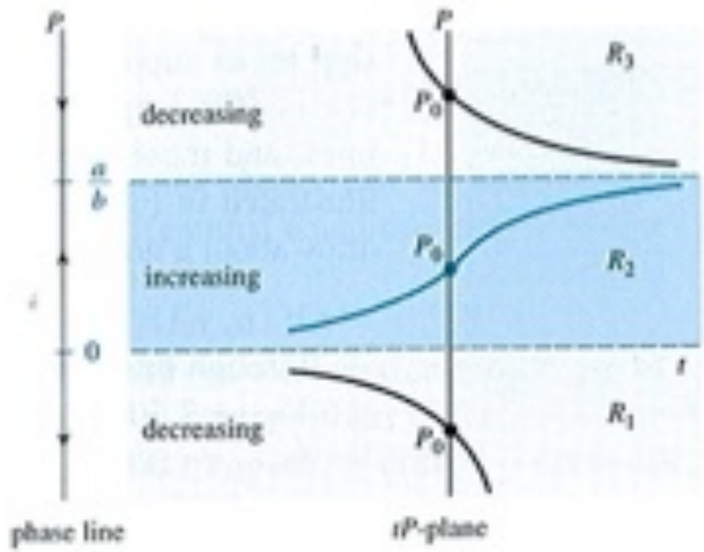
We can usually say a great deal about the solution curves of an autonomous DE even without solving it.

f in (1) is independent of x and thus we may consider it defined for any x.

f and f' are continuous functions of x on some interval I, the fundamental result of the uniqueness theorem holds in some region R in the xy-plane, and so through any point (x_0, y_0) in R passes only one solution curve of (1).

Assume that the solution of (1) possesses exactly two critical points c_1 and c_2. The graphs of the equilibrium solutions $y(x) = c_1$ and $y(x) = c_2$ are horizontal lines which partition the region R into three subregions R_1, R_2 and R_3.





Example:

$$\frac{dP}{dt} = P(a - bP)$$

where $a > 0$, $b > 0$. We have three subregions

$R_1 : -\infty < P < 0, \quad R_2 : 0 < P < a/b, \quad R_3 : a/b < P < \infty;$

(i) $P_0 < 0$: $P(t)$ is bounded from above, $P(t)$ is decreasing, $P(t) \to 0$ as $t \to -\infty$.

(ii) $0 < P_0 < a/b$: $P(t)$ is bounded from both below and above, $P(t)$ is increasing, $P(t) \to 0$ as $t \to -\infty$ and $P(t) \to a/b$ as $t \to \infty$.

(iii) $P_0 > a/b$: $P(t)$ is bounded from below, $P(t)$ is decreasing, $P(t) \to a/b$ as $t \to \infty$.

Example:

$$\frac{dy}{dx} = (y-1)^2$$

has the single critical point 1.

A solution $y(x)$ is an increasing function in both subregions $-\infty < y < 1$ and $1 < y < \infty$, where $-\infty < x < \infty$.

For an initial condition $y(0) = y_0 < 1$, a solution $y(x)$ is increasing and bounded above by 1, so $y(x) \to 1$ as $x \to \infty$.

For $y(0) = y_0 > 1$ a solution $y(x)$ is increasing and unbounded.

$y(x) = 1 - 1/(x + c)$ is a one-parameter family of solutions of the DE

$$\frac{dy}{dx} = (y - 1)^2$$

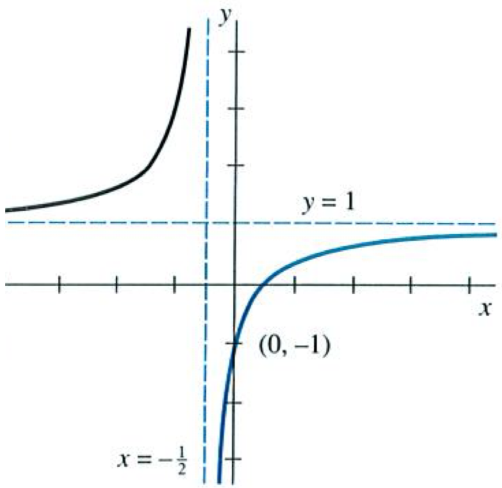
The initial condition determines the value of c:

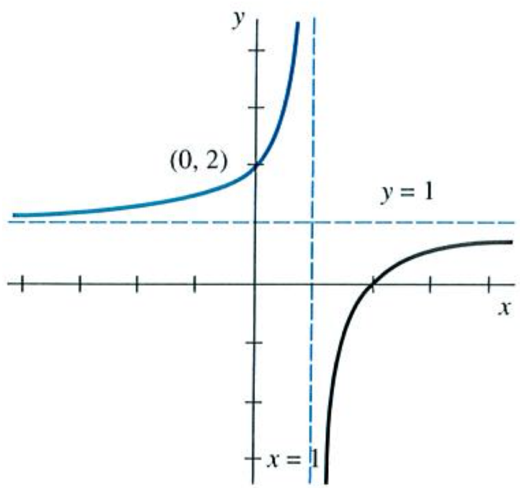
(1) $y(0) = -1 < 1$ then $c = 1/2$ and so $y(x) = 1 - 1/(x + 1/2)$

$x = -1/2$ is the vertical asymptote and $y(x) \to -\infty$ as $x \to -1/2$ from the right.

(2) $y(0) = 2 > 1$, we get $c = -1$ and $y(x) = 1 - 1/(x - 1)$.

This function has a vertical asymptote at $x = 1$ and thus $y \to \infty$ as $x \to 1$.



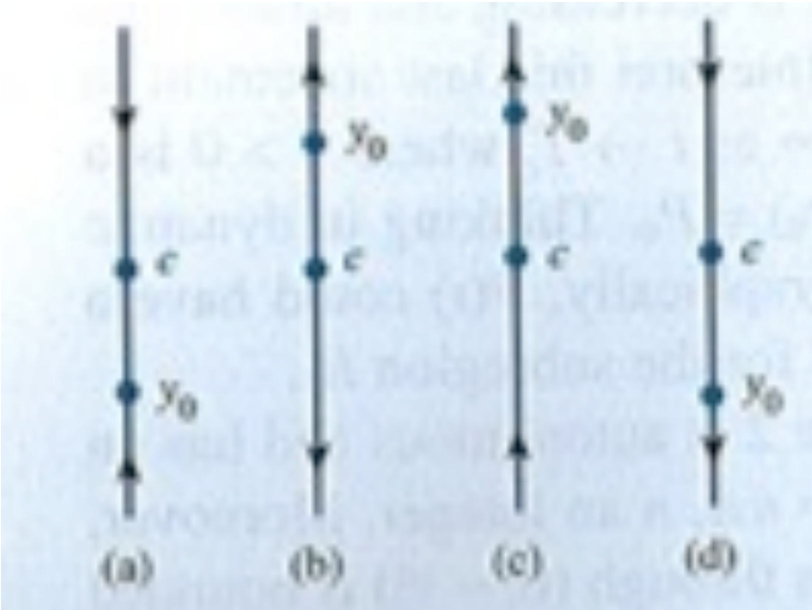


Attractors and repellers

The critical point c to which the solutions asymptotically converge from both sides is said to be **asymptotically stable**. c is referred to as an **attractor**.

The critical point c from which the solutions asymptotically diverge to both sides is said to be **unstable**. c is referred to as an **repeller**.

There are also critical points which are
neither attractors nor repellers;
they are attracted from one
side of the critical point and
repelled from the other side;
we say that c is **semistable**.



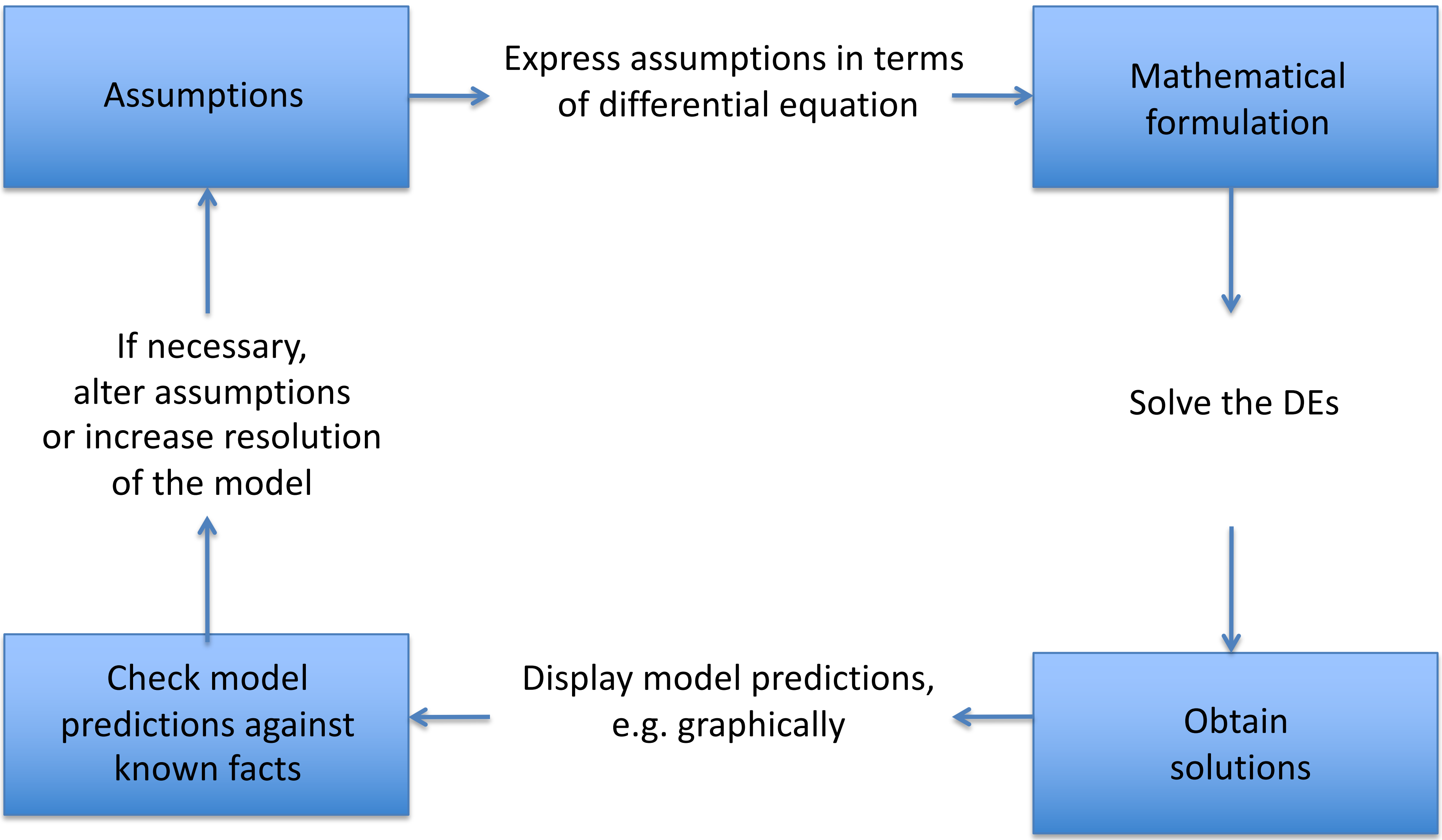
Frank and Ernest

Mathematical model

is the mathematical descriptions of a system or a phenomenon. Construction:

- identifying variables, including specifying the **level of resolution**;
- making a set of reasonable assumptions or hypotheses about the system, including empirical laws that are applicable; these often involve a rate of change of one or more variables and thus differential equation.
- trying to solve the model, and if possible, verifying, improving: increasing resolution, making alternative assumptions etc.

A mathematical model of a physical system will often involve time. A solution of the model then gives the **state of the system**, the values of the dependent variable(s), at a time t, allowing us to describe the system in the past, present and future.

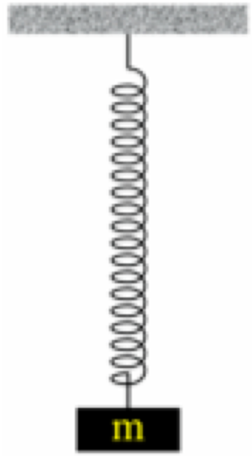

Assumptions
Express assumptions in terms of differential equation
Mathematical formulation
Solve the DEs
Obtain solutions
Display model predictions, e.g. graphically
Check model predictions against known facts
If necessary, alter assumptions or increase resolution of the model

Examples of ordinary differential equations

(1) **Spring-mass problem**

Newton's law

$$F = ma = m\frac{\mathrm{d}v}{\mathrm{d}t} = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$$



Hook's law

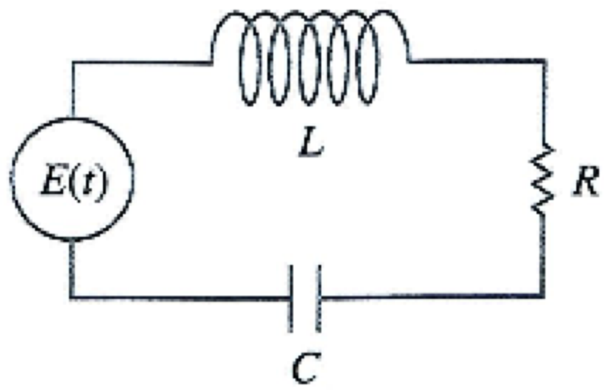
$$F = -kx$$

By putting these two laws together we get the desired ODE

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x \quad = \quad 0$$

where we introduced the angular frequency of oscillation $\omega = \sqrt{k/m}$.

(2) **RLC circuit**



$i(t)$ - the current in a circuit at time t

$q(t)$ - the charge on the capacitor at time t

L - inductance

C - capacitance

R - resistance

According to **Kirchhoff's second law**, the impressed voltage $E(t)$ must equal to the sum of the voltage drops in the loop.

$$V_L + V_C + V_R = E(t)$$

Inductor

$$V_L = L\frac{\mathrm{d}i}{\mathrm{d}t} = L\frac{\mathrm{d}^2q}{\mathrm{d}t^2}$$

Capacitor

$$V_C = \frac{q}{C}$$

Resistor

$$V_R = Ri = R\frac{\mathrm{d}q}{\mathrm{d}t}$$

RLC circuit

$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q = E(t)$$