Special functions

- Legendre’s equations

Legendre equation

The differential equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0.$$

is called **Legendre's equation of order n**.

Its solutions are called **Legendre functions**.

We will only consider the case when n is a non-negative integer and we will seek series solutions about $x = 0$.

Solution:

Since $x = 0$ is an ordinary point of Legendre's equation, we substitute the series $y = \sum_{k=0}^{\infty} c_k x^k$, shift summation indices and combine series to get

$$\begin{aligned} 1 - x^2)y'' - 2xy' + n(n+1)y &= [n(n+1)c_0 + 2c_2] + [(n-1)(n+2)c_1 + 6c_3]x \\ &\quad + \sum_{j=2}^{\infty} [(j+2)(j+1)c_{j+2} + (n-j)(n+j+1)c_j]x^j = 0 \end{aligned}$$

which implies

$$\begin{aligned} n(n+1)c_0 + 2c_2 &= 0 \\ (n-1)(n+2)c_1 + 6c_3 &= 0 \\ (j+2)(j+1)c_{j+2} + (n-j)(n+j+1)c_j &= 0 \end{aligned}$$

or

$$
\begin{aligned}
c_2 &= -\frac{n(n+1)}{2!}c_0 \\
c_3 &= -\frac{(n-1)(n+2)}{3!}c_1 \\
c_{j+2} &= -\frac{(n-j)(n+j+1)}{(j+2)(j+1)}c_j, \qquad j = 2, 3, 4, \ldots\ .
\end{aligned}
$$

For $j = 2, 3, 4, \ldots,$ the recurrence relation above gives explicitly

$$\begin{aligned}
c_4 &= -\frac{(n-2)(n+3)}{4.3}c_2 = \frac{(n-2)n(n+1)(n+3)}{4!}c_0 \\
c_5 &= -\frac{(n-3)(n+4)}{5.4}c_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}c_1 \\
c_6 &= -\frac{(n-4)(n+5)}{6.5}c_4 = -\frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}c_0 \\
c_7 &= -\frac{(n-5)(n+6)}{7.6}c_5 = -\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}c_1
\end{aligned}$$

and so on.

Thus for at least $|x| < 1$, we obtain two linearly independent solutions:

$$\begin{aligned} y_1(x) &= c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 \right. \\ &\quad \left. - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6 + \ldots\right] \\ y_2(x) &= c_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 \right. \\ &\quad \left. - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7 + \ldots\right] \end{aligned}$$

Notice that if n is an even integer, the first series terminates, whereas $y_2(x)$ is an infinite series.

For example, if $n = 4$, then

$$y_1(x) = c_0\left[1 - \frac{4.5}{2!}x^2 + \frac{2.4.5.7}{4!}x^4\right] = c_0\left[1 - 10x^2 + \frac{35}{3}x^4\right].$$

Similarly, when n is an odd integer, the series for $y_2(x)$ terminates with x^n, so *when n is a nonnegative integer, we obtain an nth degree polynomial solution* of Legendre's equation.

Since a constant multiple of a solution of Legendre's equation is also a solution, it is usual to choose specific values for c_0 and c_1, depending whether n is an even or odd positive integer: for $n = 0$, we choose $c_0 = 1$ and for

$$\begin{array}{ll} n = 0 & c_0 = 1, \\ n = 2, 4, 6, \ldots & c_0 = (-1)^{n/2} \frac{1.3...(n-1)}{2.4...n}, \end{array}$$

$$\begin{array}{ll} n = 1 & c_1 = 1, \\ n = 3, 5, 7, \ldots & c_1 = (-1)^{(n-1)/2} \frac{1.3...n}{2.4...(n-1)}. \end{array}$$

For example, if $n = 4$, we have

$$y_1(x) = (-1)^{4/2}\frac{1.3}{2.4}\left[1 - 10x^2 + \frac{35}{3}x^4\right] = \frac{1}{8}\left(3 - 30x^2 + 35x^4\right).$$

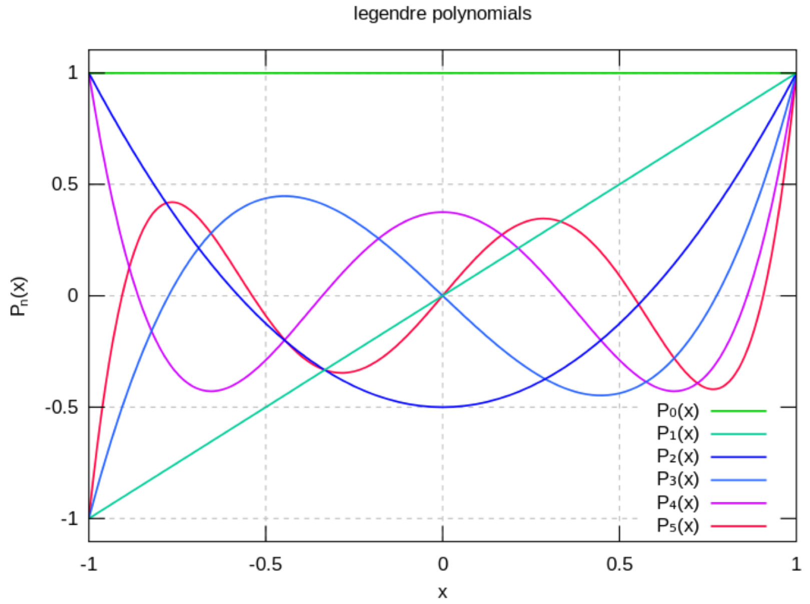
Legendre polynomials

These specific nth degree polynomial solutions are called **Legendre polynomials** $P_n(x)$. From the series $y_1(x)$ and $y_2(x)$ and the choices of c_0 and c_1 above, the first several Legendre polynomials are

$$P_0(x) = 1, \qquad P_1(x) = x,$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1), \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x),$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x).$$



The Legendre polynomials, $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)\ldots$, are particular solutions of the differential equations

$$
\begin{aligned}
n = 0: &\quad (1-x^2)y'' - 2xy' = 0 \\
n = 1: &\quad (1-x^2)y'' - 2xy' + 2y = 0 \\
n = 2: &\quad (1-x^2)y'' - 2xy' + 6y = 0 \\
n = 3: &\quad (1-x^2)y'' - 2xy' + 12y = 0 \\
\vdots &\quad \vdots
\end{aligned}
$$

Properties of Legendre polynomials

(i) $P_n(x)$ is even or odd according to whether n is even or odd respectively

$$P_n(-x) = (-1)^n P_n(x)$$

(ii)

$$P_n(1) = 1$$

(iii)

$$P_n(-1) = (-1)^n$$

(iv)

$$P_n(0) = 0, n \text{ odd}$$

(v)

$$P_n'(0) = 0, n \text{ even}$$

Recurrence relation

relates Legendre polynomials of different degrees

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0, \quad k = 1, 2, 3, \ldots .$$

For example, we can express $P_6(x)$ in terms of $P_4(x)$ and $P_5(x)$.

Rodrigues' formula

can generate the Legendre polynomials by differentiation

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n, \quad n = 0, 1, 2, \ldots \ .$$

Remarks:

In a more general setting, n can represent any real number.

If n is not a nonnegative integer, then both Legendre functions $y_1(x)$ and $y_2(x)$ are infinite series convergent on the open interval $(-1, 1)$ and divergent at $x = \pm 1$.

If n is a nonnegative integer, then one of the Legendre functions is a polynomial and the other is an infinite series convergent for $-1 < x < 1$.

Legendre's equation possesses solutions that are bounded on the closed interval $[-1, 1]$ only in the case that $n = 0, 1, 2, \ldots,$ and the only Legendre functions that are bounded on the closed interval $[-1, 1]$ are the Legendre polynomials $P_n(x)$ and their constant multiples.

Associated Legendre's equation

$$(1-x^2)v'' - 2xv' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]v = 0$$

has regular solutions in terms of **associated Legendre functions**

$$v = P_n^m(x) = (1-x^2)^{m/2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}P_n(x) = \frac{1}{2^n n!}(1-x^2)^{m/2}\frac{\mathrm{d}^{m+n}}{\mathrm{d}x^{m+n}}(x^2-1)^n, \quad -n \le m \le n$$

With the identification $x = \cos\theta$ and after normalization, these are related to spherical harmonic functions

$$Y_n^m(\theta,\varphi) = (-1)^m\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}P_n^m(\cos\theta)e^{im\varphi}.$$

which are orthonormal over the spherical surface. They play very important role in quantum theory as eigenstates of certain operators relevant to angular momentum.

Addition theorem for spherical harmonics

Consider two different directions in space defined by the angles (θ_1, φ_1) and (θ_2, φ_2) in spherical coordinate system and separated by an angle γ. They satisfy the following trigonometric identity

$$\cos\gamma = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\varphi_1 - \varphi_2).$$

The addition theorem then asserts that

$$P_n(\cos\gamma) = \frac{4\pi}{2n+1} \sum_{m=-n}^{m=n} (-1)^m Y_n^m(\theta_1, \varphi_1) Y_n^{-m}(\theta_2, \varphi_2)$$

or equivalently

$$P_n(\cos\gamma) = \frac{4\pi}{2n+1} \sum_{m=-n}^{m=n} Y_n^m(\theta_1, \varphi_1) Y_n^{m*}(\theta_2, \varphi_2).$$

Hermite polynomials

play a prominent role in quantum theory of harmonic oscillator.

They can be defined using the following relation

$$H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(e^{-x^2} \right)$$

and satisfy the recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

and

$$H_n'(x) = 2nH_{n-1}(x).$$

Examples:

$$
\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= 2x \\
H_2(x) &= 4x^2 - 2 \\
H_3(x) &= 8x^3 - 12x \\
H_4(x) &= 16x^4 - 48x^2 + 12 \\
H_5(x) &= 32x^5 - 160x^3 + 120x \\
&\vdots
\end{aligned}
$$

The solutions of the differential equation for a quantum mechanical simple harmonic oscillator

$$\varphi_n(x)'' + (2n + 1 - x^2)\varphi_n(x) = 0$$

are given in terms of the Hermite polynomials as follows

$$\varphi_n(x) = e^{-x^2/2} H_n(x)$$

The equation above is self-adjoint and its solutions satisfy the orthogonality (with the weighting function)

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2}\, dx = 0, \quad m \neq n.$$

Laguerre differential equation

$$xy''(x) + (1 - x)y'(x) + ny(x) = 0$$

has solutions given in terms of the Laguerre polynomials

$$L_n(x) = \frac{e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n e^{-x}), \quad \text{integral } n.$$

They satisfy the recurrence relations

$$\begin{aligned} (n+1)L_{n+1}(x) &= (2n + 1 - x)L_n(x) - nL_n(x), \\ xL_n'(x) &= nL_n(x) - nL_{n-1}(x). \end{aligned}$$

Examples:

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= -x + 1 \\ 2!L_2(x) &= x^2 - 4x + 2 \end{aligned}$$

The most important application is quantum theory of the hydrogen atom.

Hypergeometric equation

$$x(1-x)y''(x) + [c - (a+b+1)x]y'(x) - aby(x) = 0$$

was introduced as a second order ODE with regular singularities at $x = 0, 1, \infty$.
One solution is given by so called hypergeometric series

$$y(x) = {}_2F_1(a, b, c; x) = 1 + \frac{a.b}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \ldots$$

where $c \neq 0, -1, -2, -3, \ldots$. The range of convergence is $|x| < 1$ and $x = 1$ for $c > a+b$, and $x = -1$ for $c > a+b-1$.
The hypergeometric functions can be rewritten

$${}_2F_1(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}$$

using Pochhammer symbol $(a)_n = a(a+1)(a+2)\ldots(a+n-1) = \frac{(a+n-1)!}{(a-1)!}$, and $(a)_0 = 1$.