# Series Solutions of Linear Differential Equations

## - solutions about singular points

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**Solutions about singular points**

Example:
Consider the differential equations

$$y'' + xy = 0, \qquad xy'' + y = 0$$

We know how to find two distinct solutions of the first equation around $x = 0$ because this point is an *ordinary point*.

However, finding two *infinite series* solutions of the second equation about the point $x = 0$ is a more difficult task as this point is a *singular point* of the equation.

A singular point $x = x_0$ of a linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

is further classified as either regular or irregular. Consider again the standard form of the equation:

$$y'' + P(x)y' + Q(x)y = 0.$$

**Definition**

A singular point $x_0$ is said to be a **regular singular point** of the differential equation above if the functions $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2Q(x)$ are both analytic at $x = x_0$. A singular point that is not regular is said to be an **irregular singular point** of the equation.

## Polynomial coefficients

We are primarily interested in linear equations

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

where the coefficients are $a_2(x)$, $a_1(x)$, and $a_0(x)$ are polynomials with no common factors.

If $a_2(x_0) = 0$, then $x = x_0$ is a singular point of the equation since at least one of the rational functions $P(x) = a_1(x)/a_2(x)$ and $Q(x) = a_0(x)/a_2(x)$ fails to be analytic at that point.

Since $a_2(x)$ is a polynomial and $x_0$ is one of its zeros, it follows that $x - x_0$ is a factor of $a_2(x)$. This means that after $a_1(x)/a_2(x)$ and $a_0(x)/a_2(x)$ are reduced to lowest term, the factor $x - x_0$ must remain, to some positive integer power, in one or both denominators.

Suppose that $x = x_0$ is a singular point of the equation but that both the functions $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are analytic at $x_0$. Then multiplying $P(x)$ by $x - x_0$ and $Q(x)$ by $(x - x_0)^2$ cancels the term $x - x_0$ in the denominators:

**If $x - x_0$ appears at most to the first power in the denominator of $P(x)$ and at most to the second power in the denominator of $Q(x)$, then $x = x_0$ is a regular singular point.**

Moreover, if $x = x_0$ is a regular singular point and we multiply the differential equation

$$y'' + P(x)y' + Q(x)y = 0.$$

by $(x - x_0)^2$, then the original equation can be put into the form:

$$(x - x_0)^2 y'' + (x - x_0)p(x)y' + q(x)y = 0$$

where $p$ and $q$ are analytic at $x = x_0$.

**Example 1:** Classification of singular points

The points $x = 2$ and $x = -2$ are singular points of the equation

$$(x^2 - 4)^2 y'' + 3(x - 2)y' + 5y = 0.$$

After dividing the equation by $(x^2 - 4)^2 = (x - 2)^2(x + 2)^2$ and reducing the coefficients to the lowest term, we get

$$P(x) = \frac{3}{(x - 2)(x + 2)^2}, \qquad Q(x) = \frac{5}{(x - 2)^2(x + 2)^2}.$$

We can now test $P(x)$ and $Q(x)$ at each singular point.

**i)** $x = 2$:

In order for $x = 2$ to be a regular singular point, the factor $x - 2$ can appear at most to the first power in the denominator of $P(x)$ and at most to the second power in the denominator of $Q(x)$. Both of these conditions are satisfied, so $x = 2$ is a regular singular point.

Alternatively, both rational functions

$$p(x) = (x-2)P(x) = \frac{3}{(x+2)^2}, \qquad q(x) = (x-2)^2 Q(x) = \frac{5}{(x+2)^2}.$$

are analytic at $x = 2$.

In this case, we can rewrite the differential equation as

$$(x-2)^2 y'' + (x-2)\frac{3}{(x+2)^2} y' + \frac{5}{(x+2)^2} y = 0.$$

**ii)** $x = -2$:

Since the factor $x - (-2) = x + 2$ appears to the second power in the denominator of $P(x)$, the point $x = -2$ is an irregular singular point of the equation.

Equivalently, the function

$$p(x) = (x + 2)P(x) = \frac{3}{(x - 2)(x + 2)}$$

is not analytic at $x = -2$.

More examples:

A) $x = 0$ is an irregular singular point of

$$x^3 y'' - 2xy' + 8y = 0$$

by inspection of denominators of $P(x) = -2/x^2$ and $Q(x) = 8/x^3$.

B) $x = 0$ is a regular singular point of

$$xy'' - 2xy' + 8y = 0$$

since $x - 0$ and $(x - 0)^2$ does not appear in the denominators of $P(x) = -2$ and $Q(x) = 8/x$ respectively.

For a singular point $x = x_0$, any nonnegative power of $x - x_0$ less than one (i.e. zero), and any nonnegative power less than two inn the denominators of $P(x)$ and $Q(x)$ respectively, imply that $x_0$ is a regular singular point.

A singular point can be a complex number:

C) $x = 3i$ and $x = -3i$ are two regular singular points of

$$(x^2 + 9)y'' - 3xy' + (1 - x)y = 0.$$

D) Any second order Cauchy-Euler equation

$$ax^2y'' + bxy' + cy = 0$$

with $a, b, c \in \mathbb{R}$, has a regular singular point at $x = 0$.

For example, the two solutions of the Cauchy-Euler equation

$$x^2y'' - 3xy' + 4y = 0$$

on the interval $(0, \infty)$ are $y_1 = x^2$ and $y_2 = x^2 \ln x$.

If we attempt to find a power series solution about the regular singular point $x = 0$, i$y = \sum_{n=0}^{\infty} c_n x^n$ , we would obtain only the polynomial solution $y = x^2$.

This is because $\ln x$ and thus also $y_2 = x^2 \ln x$ are not analytic at $x = 0$, that is, the function $y_2$ does not possess a Taylor series expansion centered at $x = 0$.

**Method of Frobenius**

to solve the differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

about a regular singular point.

**Frobenius' Theorem:**

If $x = x_0$ is a regular singular point of the differential equation above, then there exists at least one nonzero solution of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n \, (x - x_0)^n = \sum_{n=0}^{\infty} c_n \, (x - x_0)^{n+r},$$

where the number $r$ is a constant to be determined. The series will converge at least on some interval defined by $0 < x - x_0 < R$.

Remarks:

There is no assurance that two series solutions of the type indicated by Frobenius Theorem can be found.

The method of Frobenius is similar to the method of undetermined series coefficients: we substitute

$$y = \sum_{n=0}^{\infty} c_n \left(x - x_0\right)^{n+r},$$

into the given differential equation and determine the unknown coefficients $c_n$ by a recursion relation.

However, we first need to determine the unknown exponent $r$. If $r$ is a number that is not a nonnegative integer, then the corresponding solution is not a power series.

**Example 2:** Two series solutions

Because $x = 0$ is a regular singular point of the differential equation

$$3xy'' + y' - y = 0,$$

we try to find the solution of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$. Since

$$y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

we get

$$\begin{aligned}
3xy'' + y' - y &= 3\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= \sum_{n=0}^{\infty}(n+r)(3n+3r-2)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= x^r\left[r(3r-2)c_0 x^{-1} + \sum_{n=1}^{\infty}(n+r)(3n+3r-2)c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n\right] \\
&= x^r\left[r(3r-2)c_0 x^{-1} + \sum_{k=0}^{\infty}[(k+r+1)(3k+3r+1)c_{k+1} - c_k]x^k\right] = 0,
\end{aligned}$$

which implies $r(3r-2)c_0 = 0$ and

$$(k+r+1)(3k+3r+1)c_{k+1} - c_k = 0, \quad k = 0, 1, 2, \dots .$$

Since there is no reason to consider $c_0 = 0$ we must have

$$r(3r-2) = 0$$

and

$$c_{k+1} = \frac{c_k}{(k+r+1)(3k+3r+1)}, \quad k = 0, 1, 2, \dots .$$

The roots of the quadratic equation above are $r_1 = \frac{2}{3}$ and $r_2 = 0$ which then give two different recurrence relations:

$$\begin{aligned}
r_1 &= \frac{2}{3}, \quad c_{k+1} = \frac{c_k}{(3k+5)(k+1)}, \quad k = 0, 1, 2, \dots \\
r_2 &= 0, \quad c_{k+1} = \frac{c_k}{(k+1)(3k+1)}, \quad k = 0, 1, 2, \dots .
\end{aligned}$$

We find explicitly from

$$r_1 = \frac{2}{3}, \qquad c_{k+1} = \frac{c_k}{(3k+5)(k+1)}, \quad k = 0, 1, 2, \ldots$$

the following coefficients:

$$\begin{aligned}
c_1 &= \frac{c_0}{5.1} \\
c_2 &= \frac{c_1}{8.2} = \frac{c_0}{2!\,5.8} \\
c_3 &= \frac{c_2}{11.3} = \frac{c_0}{3!\,5.8.11} \\
c_4 &= \frac{c_3}{14.4} = \frac{c_0}{4!\,5.8.11.14} \\
&\vdots \\
c_n &= \frac{c_0}{n!\,5.8.11\ldots(3n+2)}.
\end{aligned}$$

And we find explicitly from

$$r_2 \quad = \quad 0, \qquad c_{k+1} = \frac{c_k}{(k+1)(3k+1)}, \quad k = 0, 1, 2, \ldots$$

the coefficients:

$$\begin{aligned}
c_1 &= \frac{c_0}{1.1} \\
c_2 &= \frac{c_1}{2.4} = \frac{c_0}{2!\ 1.4} \\
c_3 &= \frac{c_2}{3.7} = \frac{c_0}{3!\ 1.4.7} \\
c_4 &= \frac{c_3}{4.10} = \frac{c_0}{4!\ 1.4.7.10} \\
&\vdots \\
c_n &= \frac{c_0}{n!\ 1.4.7\ldots(3n-2)}.
\end{aligned}$$

The both sets of coefficients are different but contain the *same* multiple of $c_0$. Omitting this, the series solutions are

$$
\begin{aligned}
y_1(x) &= x^{2/3}\left[1+\sum_{n=1}^{\infty}\frac{1}{n!\,5.8.11\ldots(3n+2)}\,x^n\right]\\
y_2(x) &= x^{0}\;\;\left[1+\sum_{n=1}^{\infty}\frac{1}{n!\,1.4.7\ldots(3n-2)}\,x^n\right].
\end{aligned}
$$

It can be verified by the ratio test that both solutions converge for all finite $x$, $|x| < \infty$.

Also, neither series is a constant multiple of the other, so $y_1(x)$ and $y_2(x)$ are linearly independent for all $x \in \mathbb{R}$, so by superposition principle $y(x) = C_1 y_1(x) + C_2 y_2(x)$ is another solution of the differential equation $3xy'' + y' - y = 0$.

On any interval not containing the origin, such as $(0, \infty)$, this linear combination represents the general solution of this differential equation.

**Indicial equation**

The equation

$$r(3r - 2) = 0$$

is called the **indicial equation** of the problem and the values $r_1 = \frac{2}{3}$ and $r_2 = 0$ are called the **inidical roots**, or **exponents**, of the singularity $x = 0$.

In general, after substituting of $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the given differential equation and simplifying, the indicial equation is a quadratic equation in $r$ that results from equating the *total coefficient of the lowest power of $x$ to zero*.

We solve for the two values of $r$ and substitute these into a recurrence relation. Frobenius Theorem guarantees that at least one nonzero solution of the assumed series form can be found.

We can also obtain the indicial equation in advance of substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation:

If $x = 0$ is a regular singular point of the equation, then both functions $p(x) = xP(x)$ and $q(x) = x^2 Q(x)$ are analytic at $x = 0$, that is, the power series expansions

$$p(x) = xP(x) = a_0 + a_1 x + a_2 x^2 + \ldots \quad \text{and} \quad q(x) = x^2 Q(x) = b_0 + b_1 x + b_2 x^2 + \ldots$$

are valid on the interval that has a positive radius of convergence. By multiplying the differential equation in the standard form by $x^2$, we get

$$x^2 y'' + x\,[xP(x)]\,y' + \left[x^2 Q(x)\right] y = 0.$$

After substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ and the two series above, for $p(x)$ and $q(x)$, and multiplying the series, we find the indicial equation

$$r(r-1) + a_0 r + b_0 = 0.$$

**Example 3:** Two series solutions

Solve

$$2xy'' + (1+x)y' + y = 0.$$

Solution:

Substitute $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ gives

$$\begin{aligned}
2xy'' + (1+x)y' + y &= 2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} \\
&\quad + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= \sum_{n=0}^{\infty}(n+r)(2n+2r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r+1)c_n x^{n+r}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}(n+r)(2n+2r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r+1)c_n x^{n+r} \\
&= x^r\left[r(2r-1)c_0x^{-1} + \sum_{n=1}^{\infty}(n+r)(2n+2r-1)c_n x^{n-1} + \sum_{n=0}^{\infty}(n+r+1)c_n x^n\right] \\
&= x^r\left[r(2r-1)c_0x^{-1} + \sum_{k=0}^{\infty}[(k+r+1)(2k+2r+1)c_{k+1} + (k+r+1)c_k]x^k\right],
\end{aligned}
$$

which implies

$$
r(2r-1) = 0
$$

whose roots are $r_1 = \frac{1}{2}$ and $r_2 = 0$, and the recurrence relation

$$
(k+r+1)(2k+2r+1)c_{k+1} + (k+r+1)c_k = 0, \quad k = 0, 1, 2 \dots .
$$

For $r_1 = \frac{1}{2}$, we can divide by $k + \frac{3}{2}$ to obtain

$$c_{k+1} = \frac{-c_k}{2(k+1)}, \quad k = 0, 1, 2 \ldots$$

from which we can calculate the values of the coefficients

$$\begin{aligned}
c_1 &= \frac{-c_0}{2.1} \\
c_2 &= \frac{-c_1}{2.2} = \frac{c_0}{2^2.2!} \\
c_3 &= \frac{-c_2}{2.3} = \frac{-c_0}{2^3.3!} \\
c_4 &= \frac{-c_3}{2.4} = \frac{c_0}{2^4.4!} \\
&\vdots \\
c_n &= \frac{(-1)^n c_0}{2^n n!}.
\end{aligned}$$

For $r_2 = 0$, we obtain

$$c_{k+1} = \frac{-c_k}{2k+1}, \quad k = 0, 1, 2\ldots$$

from which we can calculate the values of the coefficients

$$\begin{aligned}
c_1 &= \frac{-c_0}{1} \\
c_2 &= \frac{-c_1}{3} = \frac{c_0}{1.3} \\
c_3 &= \frac{-c_2}{5} = \frac{-c_0}{1.3.5} \\
c_4 &= \frac{-c_3}{7} = \frac{c_0}{1.3.5.7} \\
&\vdots \\
c_n &= \frac{(-1)^n c_0}{1.3.5.7\ldots(2n-1)}.
\end{aligned}$$

Thus for the indicial root $r_1 = \frac{1}{2}$, we obtain the solution (after omitting $c_0$)

$$y_1(x) = x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!} x^n\right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{n+1/2}$$

This series converges for $x \geq 0$ but is not defined for the negative values of $x$ because of the presence of $x^{1/2}$.

For $r_2 = 0$, a second solution is

$$y_2(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{1.3.5.7\ldots(2n-1)} x^n, \quad |x| < \infty.$$

On the interval $(0, \infty)$ the general solution is

$$y(x) = C_1 y_1(x) + C_2 y_2(x).$$

**Example 4:** Only one series solution
Solve

$$xy'' + y = 0.$$

Solution:

From $xP(x) = 0$ and $x^2Q(x) = x$ and the fact that $0$ and $x$ are their own power series at $0$, we conclude that $a_0 = 0$ and $b_0 = 0$. The indicial equation

$$r(r-1) + a_0 r + b_0 = r(r-1) = 0.$$

and its roots $r_1 = 1$ and $r_2 = 0$ lead to recurrence relations which yield exactly the same set of coefficients.

In this case, the Frobenius method produces only one solution

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^{n+1} = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \dots .$$

**Three cases**

Suppose that $x = 0$ is a regular singular point of a linear second-order differential equation and that indicial roots $r_1$ and $r_2$ of the singularity are real and $r_1 \geq r_2$.

We distinguish three cases corresponding to the nature of the indicial roots:

**Case I:**
If $r_1$ and $r_2$ are distinct and do not differ by an integer, there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}.$$

This case was illustrated in Examples 2 and 3.

**Case II:**

If $r_1 - r_2 = N$ where $N$ is a positive integer, then there exist two linearly independent solutions of a linear second-order differential equation of the form

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad c_0 \neq 0, \\ y_2(x) &= C y_1(x)\, \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}. \end{aligned}$$

where $C$ is a constant that can be zero.

In this case, the second solution *may* contain a logarithm.

**Case III:**

If $r_1 = r_2$ then there always exist two linearly independent solutions of a linear second-order differential equation of the form

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad c_0 \neq 0, \\ y_2(x) &= y_1(x)\ \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}. \end{aligned}$$

In this case, a second solution *always* contains a logarithm.

**Finding the second solution**

In the **Case II**, when the difference $r_1 - r_2$ is a positive integer, we *may* or *may not* be able to find two solutions having the form

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}.$$

This is to be determined after the indicial roots are found and the recurrence relation that defines $c_n$ is carefully examined. We may be lucky enough to find two solutions that involve only powers of $x$

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r}$$

that is, $y_2(x)$ a second solution with $C = 0$.

In the Example 4, we see that the difference $r_1 - r_2 = 1$ is an integer and the Frobenius method failed to give a second series solution. In this situation, a second solution is

$$y_2(x) \quad = \quad C y_1(x) \ \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

with $C \neq 0$.

In the Case III, where the difference $r_1 - r_2 = 0$, the Frobenius method fails to give a second series solution. The second solution will always contain a logarithm and is actually with $C = 1$.
One way to obtain thus second solution with a logarithmic term is to use the fact that

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$$

is also a solution of $y'' + P(x)y' + Q(x)y = 0$ whenever $y_1(x)$ is the known solution.