Power series solutions about ordinary points

Consider the linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

which is put into the standard form by dividing by the coefficient $a_2(x)$:

$$y'' + P(x)y' + Q(x)y = 0.$$

Definition

A point x_0 is said to be an *ordinary point* of the differential equation (above) if both $P(x)$ and $Q(x)$ in the standard form are analytic at x_0. A point that is not an ordinary point is said to be a *singular point* of the equation.

Examples:

Every finite point x is an ordinary point of the equation $y'' + (e^x)y' + (\sin x)y = 0$.
The point $x = 0$ is a singular point of the equation $y'' + (e^x)y' + (\ln x)y = 0$.

Polynomial coefficients

We will be primarily interested in differential equations with polynomial coefficients.

If the coefficients $a_2(x)$, $a_1(x)$ and $a_0(x)$ are polynomials with no common factors, then both functions $P(x) = a_1(x)/a_2(x)$ and $Q(x) = a_0(x)/a_2(x)$ are rational functions and are analytic except where $a_2(x) = 0$.

Consequently, $x = x_0$ is an ordinary point of the equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

if $a_2(x_0) \neq 0$, whereas $x = x_0$ is a singular point of the equation if $a_2(x_0) = 0$.

Examples:

The equation $(x^2 - 1)y'' + 2xy' + 6y = 0$ has singular points at $x = \pm 1$. All other finite values of x are ordinary points.

The Cauchy-Euler equation $ax^2y'' + bxy' + cy = 0$ has a singular point at $x = 0$.

The equation $(x^2+1)y''+2xy'-y = 0$ has singular points at $x = \pm i$. All other (complex) values are ordinary points.

Theorem: Existence of power series solutions
If $x = x_0$ is an ordinary point of the differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,$$

we can always find two linearly independent solutions in the form of a power series centered at x_0; that is

$$y = \sum_{n=0}^{\infty} c_n \, (x - x_0)^n.$$

A series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point.

The solution of the form $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ is said to be **a solution about the ordinary point** x_0.

The distance R is the minimum value or **lower bound** for the radius of convergence.

Example:

The points $1 \pm 2i$ are singular points of $(x^2 - 2x + 5)y'' + xy' - y = 0$. Theorem guarantees, since $x = 0$ is an ordinary point of the equation, that we can find two power series solutions centered at $x = 0$. Moreover, the solutions will have the form $\sum_{n=0}^{\infty} c_n x^n$ and each will converge at least for $|x| < \sqrt{5}$ where $R = \sqrt{5}$ is the distance from $x = 0$ to either of the singular points. In fact, it turns out that one of the solution is a polynomial and thus valid for much larger values of x, specifically the entire interval $(-\infty, \infty)$.

Power series solution of homogeneous linear second-order ODE

The method of undetermined *series* coefficients:

- if $x_0 \neq 0$, change the variable $t = x - x_0$, otherwise

- substitute $\sum_{n=0}^{\infty} c_n \, x^n$ into the differential equation;

- combine series;

- equate all coefficients to the r.h.s. of the equation to determine c_n, since this is homogeneous equation all coefficients of x^k must be equated to zero (this does not mean that all coefficients of the series solutions are zero!).

Example 2

Solve $y'' + xy = 0$.

This is an example of Airy's equation which is relevant for example to diffraction of electromagnetic waves and aerodynamics.

Solution:

There are no finite singular points, so Theorem guarantees two solutions centered at $x = 0$ and convergent for $|x| < \infty$.

We assume solutions in the form of the power series

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

We substitute $y(x) = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation

$$y'' + xy = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + x\sum_{n=0}^{\infty} c_n x^n$$
$$= \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0.$$

and rewrite the last expression using a single summation

$$y'' + xy = 2c_2 + \sum_{n=3}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$
$$= 2c_2 + \sum_{k=1}^{\infty} \left[(k+1)(k+2)c_{k+2} + c_{k-1}\right] x^k = 0$$

$$y'' + xy = 2c_2 + \sum_{k=1}^{\infty} \left[(k+1)(k+2)c_{k+2} + c_{k-1}\right] x^k = 0$$

Since all coefficients of x^k must be equated to zero for each k, we conclude that $c_2 = 0$, and obtain the **recurrence relation** for c_k

$$c_{k+2} = -\frac{c_{k-1}}{(k+1)(k+2)}, \quad k = 1, 2, 3, \ldots \ .$$

The coefficients are explicitly

$$k = 1, \quad c_3 = -\frac{c_0}{2.3}$$

$$k = 2, \quad c_4 = -\frac{c_1}{3.4}$$

$$k = 3, \quad c_5 = -\frac{c_2}{4.5} = 0$$

$$k = 4, \quad c_6 = -\frac{c_3}{5.6} = \frac{c_0}{2.3.5.6}$$

$$k = 5, \quad c_7 = -\frac{c_4}{6.7} = \frac{c_1}{3.4.6.7}$$

$$k = 6, \quad c_8 = -\frac{c_5}{7.8} = 0$$

$$k = 7, \quad c_9 = -\frac{c_6}{8.9} = -\frac{c_0}{2.3.5.6.8.9}$$

$$k = 8, \quad c_{10} = -\frac{c_7}{9.10} = -\frac{c_1}{3.4.6.7.9.10}.$$

Substituting the coefficients to the original assumption

$$\begin{aligned} y &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8 + c_9 x^9 + c_1 0 x^{10} \\ &= c_0 + c_1 x + 0 - \frac{c_0}{2.3} x^3 - \frac{c_1}{3.4} x^4 + 0 + \frac{c_0}{2.3.5.6} x^6 + \frac{c_1}{3.4.6.7} x^7 + 0 \\ &\quad - \frac{c_0}{2.3.5.6.8.9} x^9 - \frac{c_1}{3.4.6.7.9.10} x^{10} + \ldots \end{aligned}$$

After grouping the terms containing c_0 and the terms containing c_1, we obtain the general solution in the form

$$y(x) = c_0 y_1(x) + c_1 y_2(x)$$

where

$$\begin{aligned} y_1(x) &= 1 + \frac{1}{2.3} x^3 + \frac{1}{2.3.5.6} x^6 + \frac{1}{2.3.5.6.8.9} x^9 + \ldots \\ y_2(x) &= x + \frac{1}{3.4} x^4 + \frac{1}{3.4.6.7} x^7 + \frac{1}{3.4.6.7.9.10} x^{10} + \ldots . \end{aligned}$$

Example 3

Solve

$$(x^2 + 1)y'' + xy' - y = 0.$$

Solution:

This differential equation has singular points at $x = \pm i$, so the power series solution centered at $x_0 = 0$ will converge at least for $|x| < 1$.

We assume solutions in the form of the power series

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

we substitute the assumed solution into the differential equation

$$\begin{aligned}
(x^2+1)y'' + xy' - y &= (x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^n \\
&= 2c_2 - c_0 + 6c_3 x + c_1 x - c_1 x + \sum_{n=2}^{\infty} n(n-1)c_n x^n \\
&\quad + \sum_{n=4}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=2}^{\infty} nc_n x^n - \sum_{n=2}^{\infty} c_n x^n \\
&= 2c_2 - c_0 + 6c_3 x + \sum_{k=2}^{\infty} \left[k(k-1)c_k + (k+2)(k+1)c_{k+2} + kc_k - c_k\right] x^k \\
&= 2c_2 - c_0 + 6c_3 x + \sum_{k=2}^{\infty} \left[(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2}\right] x^k = 0
\end{aligned}$$

From the last identity, we conclude that $2c_2 - c_0 = 0$, $6c_3 = 0$, and

$$(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2} = 0.$$

thus $c_2 = \frac{1}{2}c_0$, $c_3 = 0$, and

$$c_{k+2} = \frac{1-k}{1+k}c_k, \quad k = 2, 3, 4, \dots .$$

Substituting $k = 2, 3, 4, \ldots$ gives the following coefficients

$$
\begin{aligned}
c_4 &= -\frac{1}{4}c_2 = -\frac{1}{2.4}c_0 = -\frac{1}{2^2\ 2!}c_0 \\
c_5 &= -\frac{2}{5}c_3 = 0 \\
c_6 &= -\frac{3}{6}c_4 = \frac{3}{2.4.6}c_0 = \frac{1.3}{2^3\ 3!}c_0 \\
c_7 &= -\frac{4}{7}c_5 = 0 \\
c_8 &= -\frac{5}{8}c_6 = \frac{3.5}{2.4.6.8}c_0 = \frac{1.3.5}{2^4\ 4!}c_0 \\
c_9 &= -\frac{6}{9}c_7 = 0 \\
c_{10} &= -\frac{7}{10}c_8 = \frac{3.5.7}{2.4.6.8.10}c_0 = \frac{1.3.5.7}{2^5\ 5!}c_0 \\
\ldots
\end{aligned}
$$

We can now write the solution

$$\begin{aligned} y &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8 + c_9 x^9 + c_10 x^{10} \\ &= c_0 \left[1 + \frac{1}{2} x^2 - \frac{1}{2^2\ 2!} x^4 + \frac{1.3}{2^3\ 3!} x^6 - \frac{1.3.5}{2^4\ 4!} x^8 + \frac{1.3.5.7}{2^5\ 5!} x^{10} - \ldots \right] + c_1 x \\ &= c_0 y_1(x) + c_1 y_2(x). \end{aligned}$$

The solutions are the power series and the polynomial

$$\begin{aligned} y_1(x) &= 1 + \frac{1}{2} x^2 - \frac{1}{2^2\ 2!} x^4 + \frac{1.3}{2^3\ 3!} x^6 - \frac{1.3.5}{2^4\ 4!} x^8 + \frac{1.3.5.7}{2^5\ 5!} x^{10} - \ldots \\ &= 1 + \frac{1}{2} x^2 + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1.3.5 \ldots (2n-3)}{2^n\ n!}\ x^{2n}, \quad |x| < 1, \\ y_2(x) &= x. \end{aligned}$$

Example 4

Solve

$$y'' - (1 + x)y = 0.$$

Solution:
We substitute a solution in the form of a power series

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

into the equation and get

$$y'' - (1 + x)y = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

and we rewrite the last expression as a single summation

$$\begin{aligned} y'' - (1+x)y &= 2c_2 + \sum_{k=1}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} c_{k-1}x^k - c_0 - \sum_{k=1}^{\infty} c_k x^k = 0 \\ &= 2c_2 - c_0 + \sum_{k=1}^{\infty}\left[(k+2)(k+1)c_{k+2} - c_{k-1} - c_k\right]x^k = 0. \end{aligned}$$

We obtain $c_2 = c_0/2$ and the recurrence relation

$$c_{k+2} = \frac{c_k + c_{k-1}}{(k+1)(k+2)}, \quad k = 1, 2, 3, \ldots$$

in which the coefficients $c_3, c_4, c_5, \ldots$ are expressed in terms of both c_0 and c_1.

To simplify, we first choose $c_0 \neq 0$ and $c_1 = 0$ which yields coefficients for one solution that are expressed entirely in terms of c_0:

$$\begin{aligned}
c_2 &= \frac{1}{2}c_0 \\
c_3 &= \frac{c_1 + c_0}{2.3} = \frac{c_0}{2.3} = \frac{1}{6}c_0 \\
c_4 &= \frac{c_2 + c_1}{3.4} = \frac{c_0}{2.3.4} = \frac{1}{24}c_0 \\
c_5 &= \frac{c_3 + c_2}{4.5} = \frac{c_0}{4.5}\left[\frac{1}{6} + \frac{1}{2}\right] = \frac{1}{30}c_0 \\
\ldots
\end{aligned}$$

Next, choosing $c_0 = 0$ and $c_1 \neq 0$ leads to the other solution to be expressed in terms of c_1

$$
\begin{aligned}
c_2 &= \frac{1}{2}c_0 = 0 \\
c_3 &= \frac{c_1 + c_0}{2.3} = \frac{c_1}{2.3} = \frac{1}{6}c_1 \\
c_4 &= \frac{c_2 + c_1}{3.4} = \frac{c_1}{3.4} = \frac{1}{12}c_1 \\
c_5 &= \frac{c_3 + c_2}{4.5} = \frac{c_1}{4.5.6} = \frac{1}{120}c_1 \\
&\cdots
\end{aligned}
$$

The general solution of the equation is then $y = c_0 y_1(x) + c_1 y_2(x)$ where

$$\begin{aligned} y_1(x) &= 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + \ldots \\ y_2(x) &= x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \ldots . \end{aligned}$$

Each series converges for all finite values of x.

Example 5: ODE with non-polynomial coefficients
Solve

$$y'' + (\cos x)y = 0.$$

Solution:
The point $x = 0$ is an ordinary point of the equation as the function $\cos x$ is analytic at that point. Assuming the solution in the form $y(x) = \sum_{n=0}^{\infty} c_n x^n$ and using the Maclaurin series for $\cos x$, we get

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)\sum_{n=0}^{\infty} c_n x^n \\
=\quad & 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \cdots + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots\right)(c_0 + c_1x + c_2x^2 + c_3x^3 + \dots) \\
=\quad & 2c_2 + c_0 + (6c_3 + c_1)\,x + \left(12c_4 + c_2 - \frac{1}{2}c_0\right)x^2 + \left(20c_5 + c_3 - \frac{1}{2}c_1\right)x^3 + \cdots = 0.
\end{aligned}$$

It follows that

$$2c_2 + c_0 = 0, \quad 6c_3 + c_1 = 0, \quad 12c_4 + c_2 - \frac{1}{2}c_0 = 0, \quad 20c_5 + c_3 - \frac{1}{2}c_1 = 0,$$

which gives $c_2 = \frac{1}{2}c_0$, $c_3 = -\frac{1}{6}c_1$, $c_4 = -\frac{1}{12}c_0$, $c_5 = \frac{1}{30}c_1, \ldots$.

By grouping terms, we get the general solution $y = c_0y_1(x) + c_1y_2(x)$ where

$$\begin{aligned} y_1(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \ldots \\ y_2(x) &= x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \ldots . \end{aligned}$$

Since the differential equation has no finite singular points, both power series converge for $|x| < \infty$.