Series Solutions of Linear Differential Equations

- solutions about ordinary points

Series Solutions of Linear Differential Equations
Introduction

Most linear higher order ODEs with *variable coefficients* cannot be solved in terms of elementary functions. Instead a solution is seeked in the form of infinite series and proceeds in a manner similar to the method of undetermined coefficients.

We will first study solutions about *ordinary points*. Given a linear second order ODE

$$y'' = f(x, y, y')$$

we say that a point $x = x_0$ is an *ordinary point* if, at this point, y and y' can take on all finite values and y'' remains finite.

On the other hand, if y'' becomes infinite for any finite choice of y and y', point $x = x_0$ is called a *singular*.

Review of power series

A power series in $x - a$, or **power series centered on** a, is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots \quad .$$

Examples:

Power series centered at $a = -1$: $\sum_{n=0}^{\infty} (x+1)^n$.

Power series in x, or centered on $a = 0$: $\sum_{n=0}^{\infty} 2^{n-1} x^n = x + 2x^2 + 4x^3 + \ldots$.

Convergence

A power series

$$\sum_{n=0}^{\infty} c_n (x - a)^n$$

is **convergent** at a specified value of x if its sequence of partial sums $\{S_N(x)\}$ converges, that is, if

$$\lim_{N \to \infty} S_N(x) = \lim_{N \to \infty} \sum_{n=0}^{N} c_n (x - a)^n$$

exists.

If the limit does not exist at x, the series is said to be **divergent**.

Interval of convergence

The interval of convergence is the set of all real numbers x for which the series converges. Every power series has an interval of convergence.

Radius of convergence

Every power series has a radius of convergence R. If $R > 0$, then a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $|x-a| < R$ and diverges for $|x-a| > R$.

If the series converges only at its center a, then $R = 0$.

If the series converges for all x, then $R = \infty$.

Also, $|x-a| < R$ is equivalent to $a - R < x < a + R$. A power series may or may not converge at the endpoints $a - R$ and $a + R$.

Absolute convergence

Within its interval of convergence a power series converges absolutely, that is, if x is a number in the interval of convergence and is not an endpoint of the interval, then the series of absolute values

$$\sum_{n=0}^{\infty} \left|c_n (x-a)^n\right|$$

converges.

Ratio test

Convergence of a power series can be determined by a **ratio test**:

Suppose that $c_n \neq 0$ for all n, and that

$$\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| = |x-a| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = L.$$

If $L < 1$ the power series converges absolutely;

if $L > 1$ the series diverges; and

if $L = 1$ the test is inconclusive.

Example For the power series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n2^n}$ the ratio test gives

$$\lim_{n\to\infty}\left|\frac{(x-3)^{n+1}/2^{n+1}(n+1)}{(x-3)^n/2^n n}\right| = |x-3| \lim_{n\to\infty}\frac{n}{2(n+1)} = \frac{1}{2}|x-3|\,.$$

The series converges absolutely for $\frac{1}{2}|x-3| < 1$, or $|x-3| < 2$, or $1 < x < 5$ which is referred as an *open* interval of convergence.

The series diverges for $|x-3| > 2$, that is, for $x < 1$ and $x > 5$.

At the left endpoint $x = 1$ of the interval of convergence, the series of constants $\sum_{n=1}^{\infty}[(-1)^n/n]$ is convergent (by alternating series test).

At the right endpoint $x = 5$, the series $\sum_{n=1}^{\infty}(1/n)$ is the divergent harmonic series.

The interval of convergence of the series is $[1, 5)$ and the radius of convergence is $R = 2$.

A power series defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n \, (x-a)^n$$

whose domain is the interval of convergence of the series.

If the radius of convergence is $R > 0$, then $f(x)$ is continuous, differentiable, and integrable on the interval $(a-R, a+R)$.

Moreover, $f'(x)$ and $\int f(x)\, dx$ can be found by term-by-term differentiation of integration. Convergence at an endpoint may be either lost by differentiation or gained via integration.

If

$$y = \sum_{n=0}^{\infty} c_n x^n$$

is a power series in x, then the first two derivatives are

$$\begin{aligned} y' &= \sum_{n=0}^{\infty} c_n n x^{n-1} = \sum_{n=1}^{\infty} c_n n x^{n-1}, \\ y'' &= \sum_{n=0}^{\infty} c_n n(n-1) x^{n-2} = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}, \end{aligned}$$

where the first term in the first derivative and the first two terms in the second derivative are zero and thus are omitted.

Identity property

If

$$y = \sum_{n=0}^{\infty} c_n (x - a)^n = 0, \quad R > 0,$$

for all numbers x in the interval of convergence, then $c_n = 0$ for all n.

Analytic at a point

A function f is analytic at a point a if it can be represented by a power series in $x - a$ with a positive radius of convergence.

Examples

Functions such as e^x, $\cos x$, $\sin x$, $\ln(x-1)$ can be represented using Taylor series:

$$
\begin{aligned}
e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}
\end{aligned}
$$

for $|x| < \infty$.

These Taylor series centered at 0, called Maclaurin series, show that e^x, $\sin x$ and $\cos x$ are analytic at $x = 0$.

Arithmetic of power series

Power series can be combined through the operations of addition, multiplication and division using procedures similar to addition, multiplication and division of polynomials:

- we add coefficients of like powers of x,
- use the distributive law and collect like terms, and
- perform a long division.

Example

$$
\begin{aligned}
e^x \sin x &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} \ldots\right) \\
&= x + x^2 + \left(-\frac{1}{6} + \frac{1}{2}\right)x^3 + \left(-\frac{1}{6} + \frac{1}{6}\right)x^4 + \left(\frac{1}{120} - \frac{1}{12} + \frac{1}{24}\right)x^5 + \ldots \\
&= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \ldots \quad .
\end{aligned}
$$

Since the power series for e^x and $\sin x$ converge for $|x| < \infty$, the product series converges on the same interval.

Shifting the summation index

It is important to simplify the sum of two or more power series, each series being expressed as a summation on its own right, to an expression involving a single summation only.

Example: Write

$$\sum_{n=2}^{\infty} n(n-1)\, c_n\, x^{n-2} + \sum_{n=0}^{\infty} c_n\, x^{n+1}$$

as one power series.

Solution:

It is necessary that
- both summation indices start with the same number and that

- the powers of x in each series be "in phase", i.e. if one series starts with a multiple of, say, x to the first power, then we want the other series to start with the same power:

$$\sum_{n=2}^{\infty} n(n-1)\, c_n\, x^{n-2} + \sum_{n=0}^{\infty} c_n\, x^{n+1} = 2.1\, c_2\, x^0 + \sum_{n=3}^{\infty} n(n-1)\, c_n\, x^{n-2} + \sum_{n=0}^{\infty} c_n\, x^{n+1},$$

where both series on the r.h.s. start with the same power of x, i.e. x^1.

Now to get the same summation index, we let $k = n-2$ in the first series and $k = n+1$ in the second series:

$$2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} c_{k-1}x^k.$$

Note that k is just a "dummy" index; it is the *value* of the summation index that is important.

We can now complete the solution:

$$\sum_{n=2}^{\infty} n(n-1)\, c_n\, x^{n-2} + \sum_{n=0}^{\infty} c_n\, x^{n+1} = 2c_2 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} + c_{k-1}\right] x^k.$$