Mathematical Physics

MP205 Vibrations and Waves

Lecture 1 - 2

MP205 Vibrations and Waves

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TUTORIAL SESSIONS: starting the next week (February 5)
Option 1: **MONDAY 10:05, Hall H**
Option 2: **WEDNESDAY 14:05, Physics Hall**

	9:05	10:05	11:05	12:05	1:05	2:05	3:05	4:05	5:05
Mon		T Hall H							
Tue	L Physics Hall		L CB3						
Wed						T Physics Hall			
Thu									
Fri									

MP205 Vibrations and Waves

Syllabus:

Simple harmonic motion

Superposition of periodic motions

Forced and damped oscillations and resonance

Coupled oscillators and normal modes

Vibrations in continuous systems and Fourier analysis

Traveling and standing waves

Sound and light as transverse and longitudinal waves

Dispersion and group velocity

Boundary effects and interference

Elementary quantum mechanics

Requirements:

Examination (constitutes 80% of the total mark):

duration: 120 minutes,

requirements: answer all questions and subquestions in writing,

maximum mark: 80 points.

Continuous Assessment (20% of the total mark):

approximately 10 homework assignments & quizzes in tutorial sessions.

REFERENCES

Lecture notes:

online access: http://www.thphys.nuim.ie/Notes/MP205/

Textbooks:

A. P. French
Vibrations and Waves
The M.I.T. Introductory Physics Series
Norton

I. G. Main
Vibrations and Waves in Physics
Cambridge University Press

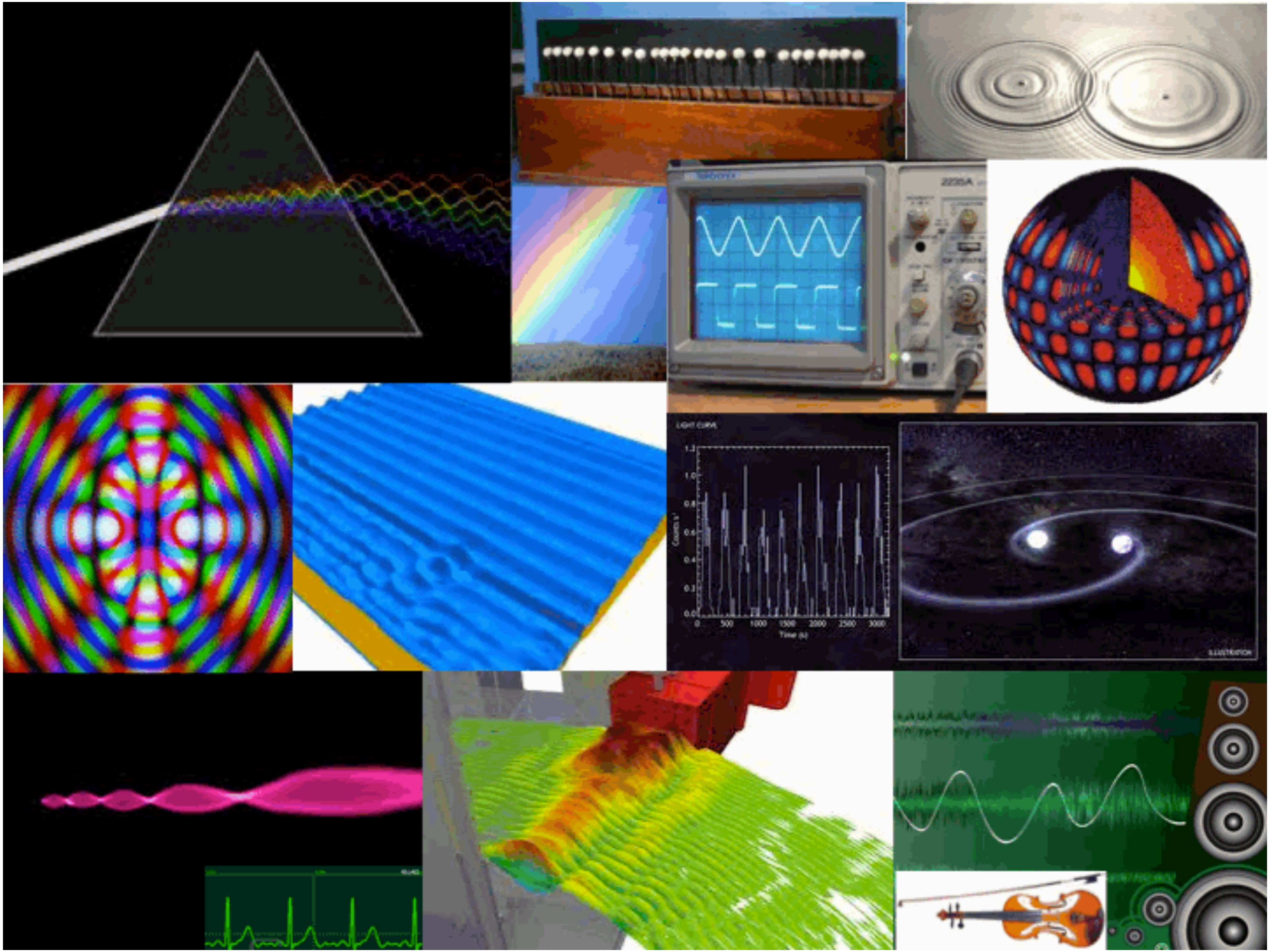
I. PERIODIC MOTIONS

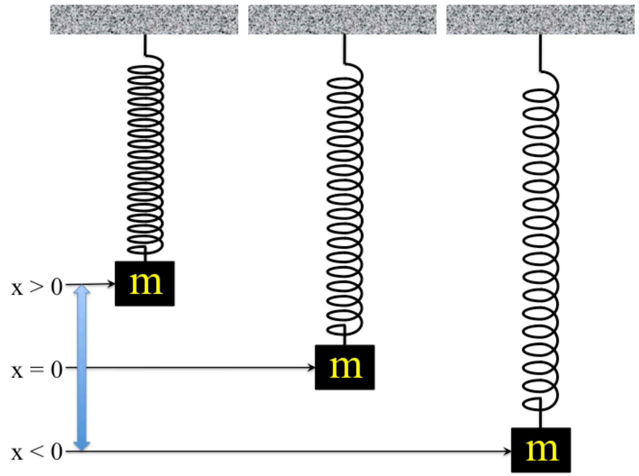
Vibrations and oscillations constitute one of the most important study in all physics as virtually every system possesses the capability for vibrations.

What these phenomena have in common is

PERIODICITY

i.e. a pattern of movement or displacement that repeats itself over and over again.


2235A
LIGHT CURVE
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
500
1000
1500
2000
2500
3000
Time (s)
ILLUSTRATION



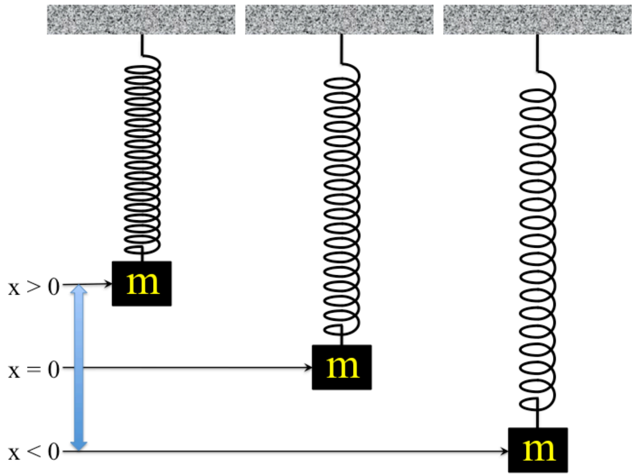
Sinusoidal vibrations

Force at a **displacement** x from equilibrium:

$$F(x) = -\left(k_1 x + k_2 x^2 + k_3 x^3 + \ldots\right)$$

Assuming the constant $k_1 >> k_2,\ k_3$ for some x, the force $F(x)$ is dominated by $-k_1 x$.

Sinusoidal vibrations



Force at a **displacement** x from equilibrium:

$$F(x) = -\left(k_1 x + k_2 x^2 + k_3 x^3 + ...\right)$$

Assuming the constant $k_1 >> k_2,\ k_3$ for some x, the force $F(x)$ is dominated by $-k_1 x$.

The equation of motion is then given by the Newton law as

$$m \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\ k_1\ x \tag{1}$$

Its solution is the **simple harmonic motion (SHM)** given by a function

$$x(t) = A\ \sin\left(\omega t + \phi_0\right) \tag{2}$$

where $\omega = \sqrt{k_1/m}$ is **the angular frequency** of the periodic motion.

Description of simple harmonic motion (SHM)

$$x(t) = A \sin(\omega t + \phi_0) \qquad\qquad \omega = \sqrt{k/m}$$

Characteristic features of SHM:

- it is confined within $x = \pm A$ where $A \geq 0$ is **the amplitude of the motion**;

- the motion has **the period** T which is the time successive occassions on which both the displacement x and the velocity $\mathrm{v} = dx/dt$ repeat themeselves:

  $$x(t) = A \sin(\omega t + \phi_0) \qquad \Rightarrow \qquad \omega(t+T) + \phi_0 = \omega t + \phi_0 + 2\pi$$

  whence

  $$T = \frac{2\pi}{\omega} \tag{3}$$

- The situation at $t = 0$ (or any other time) is completely specified by the values of x and $\mathrm{v} = dx/dt$ at that instant.

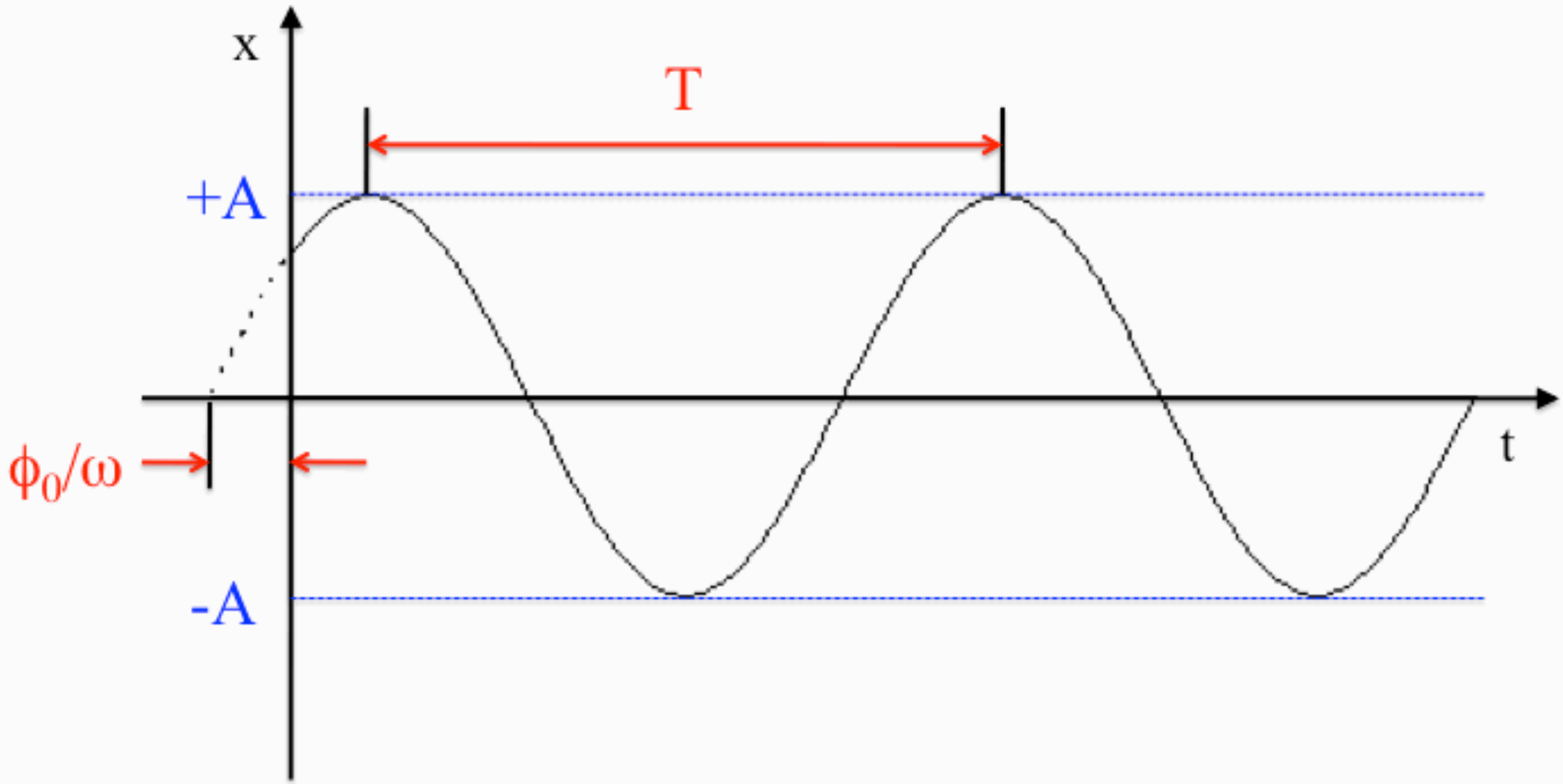
  For a particular time $t = 0$, let denote $x_0 = x|_{t=0}$ and $\mathrm{v}_0 = dx/dt|_{t=0}$ then

$$\begin{aligned} x_0 &= A \ \sin\phi_0 \qquad\qquad (\Leftarrow x(t) = A\sin(\omega t + \phi_0)) \\ \mathrm{v}_0 &= \omega A \ \cos\phi_0 \end{aligned}$$

  which implies that

$$A = \left[x_0^2 + \left(\frac{\mathrm{v}_0}{\omega}\right)^2\right]^{1/2} \tag{4}$$

$$\phi_0 = tan^{-1}\left(\frac{\omega\ x_0}{\mathrm{v}_0}\right) \tag{5}$$


x
T
+A
-A
t
ϕ_0/ω

Remarks:

- the value of the angular frequency ω is assumed o be known independently;

- the simple harmonic motion of an actual physical system must be long-continued (steady-state vibration) for the equation $x(t) = A \sin(\omega t + \phi_0)$ to provide an acceptable description.

$A \sin(\omega t)$ $\omega = 8\pi/5 \text{ rad.s}^{-1}$

$A \sin(2\omega t)$

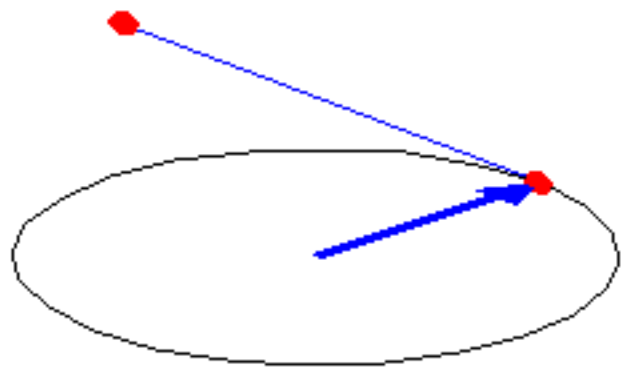
$A \sin(\omega t/2)$

$A \sin(\omega t + \phi_0)$ $\phi_0 = \pi/4$

$2A \sin(\omega t)$

Rotating vector representation

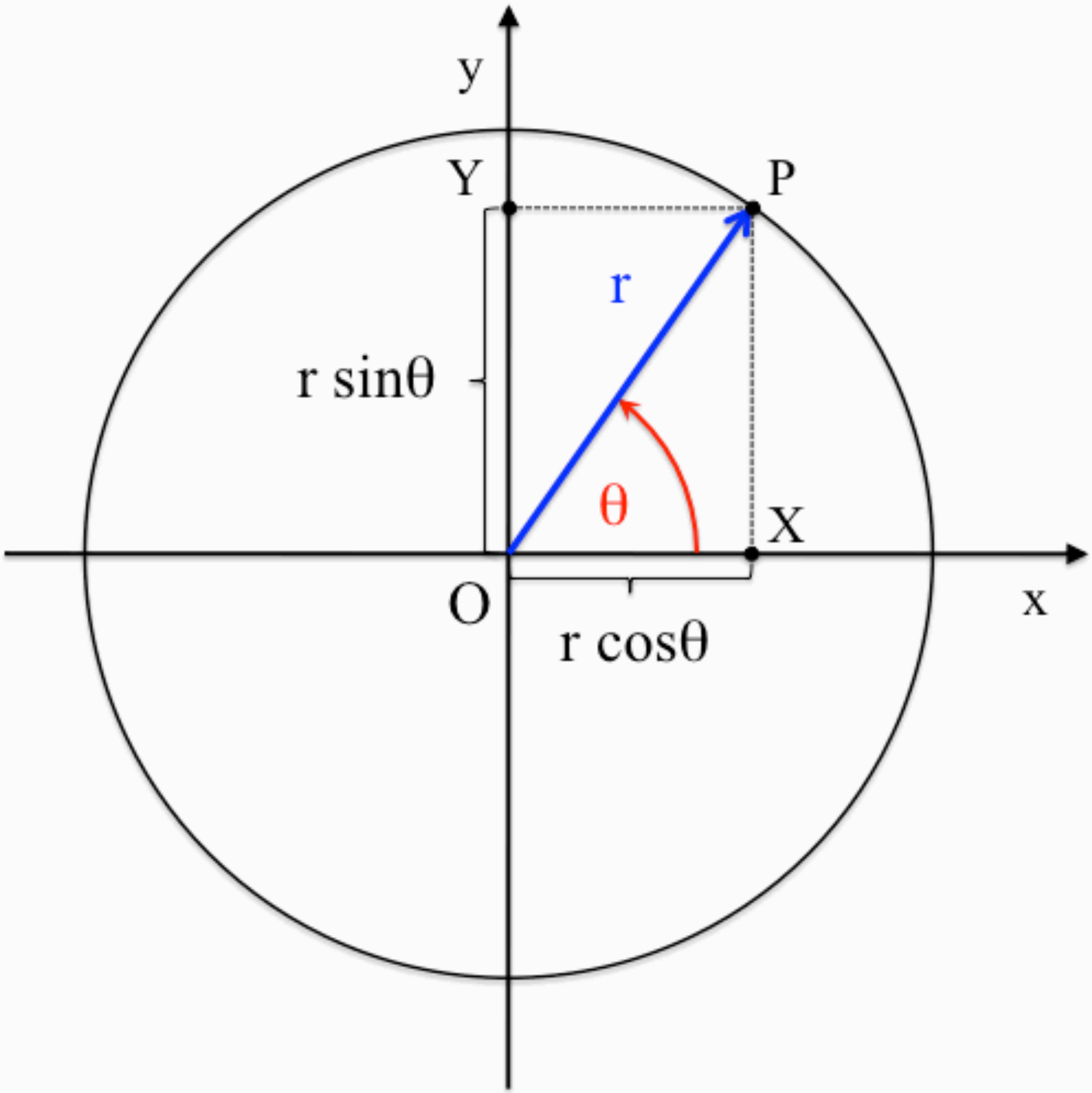
SHM can be imagined as **the geometric projection of uniform circular motion**.



It can thus be described equally well in terms of sine and cosine functions:

$$A\ \sin(\omega t + \phi_0) \;=\; A\ \cos(\omega t + \alpha) = A\ \sin\left(\omega t + \alpha + \frac{\pi}{2}\right) \;\Rightarrow\; \phi_0 = \alpha + \frac{\pi}{2}$$

We will use the cosine form which connects with the geometric interpretation.


y
Y
P
r
r sinθ
θ
X
O
x
r cosθ

Rotating vectors and complex numbers

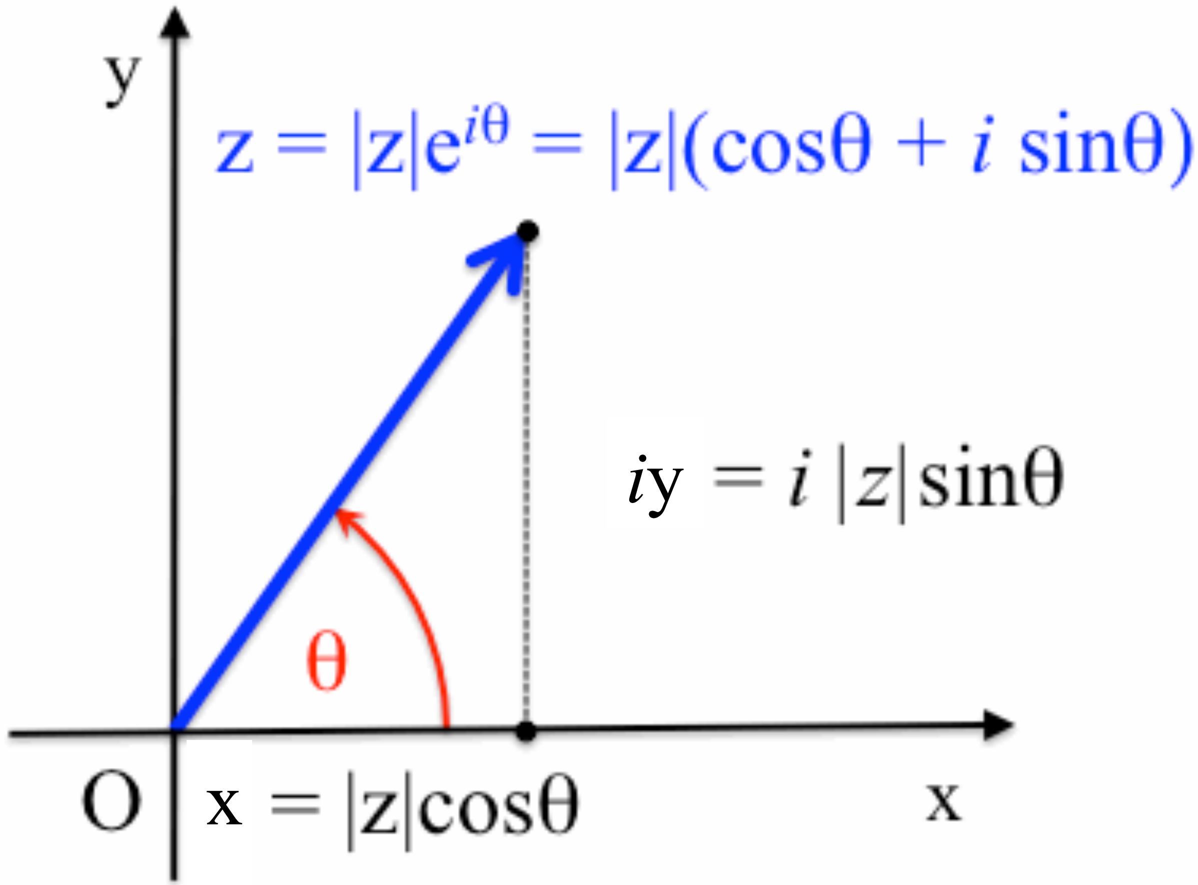
The circular motion defines SHM (with A and ω) along any straight line in the plane of the circle

$$
\begin{aligned}
x(t) &= A\ \cos(\omega t + \alpha) = r\ \cos\theta \\
y(t) &= A\ \sin(\omega t + \alpha) = r\ \sin\theta
\end{aligned}
$$

where r and θ are polar coordinates.
We can write the vector $\vec{r} = (x(t), y(t))$ in complex notation, using $i = \sqrt{-1}$, as

$$x(t)\ +\ i\ y(t) = z(t) \tag{6}$$

The quantity x represents SHM while the quantity y is physically irrelevant.


y
z = |z|e^{iθ} = |z|(cosθ + i sinθ)
iy = i |z|sinθ
θ
O
x = |z|cosθ
x

Introducing the complex exponentials

The Taylor series expansion $f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!}\, f''(0) + ...$ of sine and cosine functions

$$
\begin{aligned}
\cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + ... \\
\sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ...
\end{aligned}
$$

allow us to rewrite $\cos\theta + i\sin\theta$ as

$$
\begin{aligned}
\cos\theta + i\sin\theta &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + ... = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + ... \\
\cos\theta + i\sin\theta &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = e^{i\theta} \qquad (7)
\end{aligned}
$$

Using the complex exponentials is convenient

a) displacement

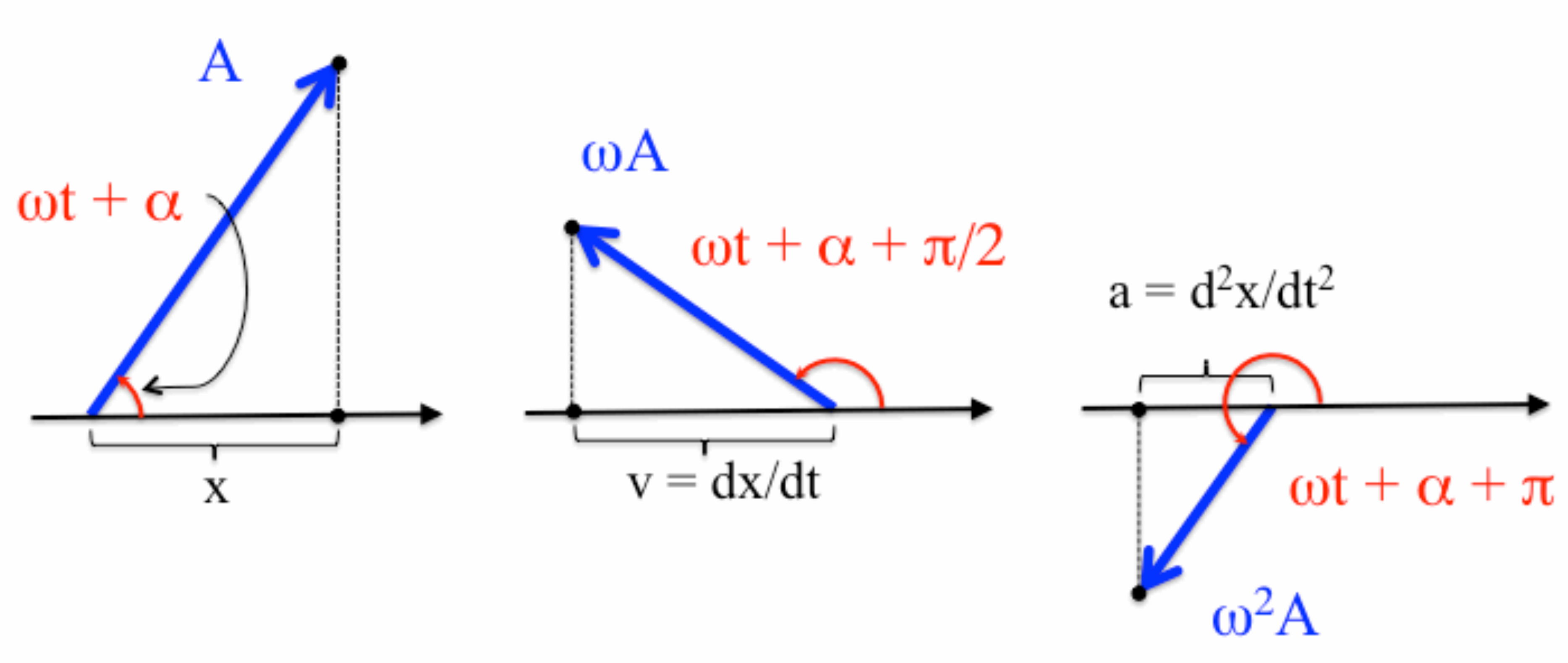
$$
\begin{aligned}
x(t) &= A\ \cos(\omega t + \alpha) = Re(z(t)) \\
z(t) &= A\ \cos(\omega t + \alpha) + iA\ \sin(\omega t + \alpha) = A\ e^{i(\omega t + \alpha)}
\end{aligned}
\tag{8}
$$

b) velocity

$$
\begin{aligned}
\mathrm{v}(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}[A\ \cos(\omega t + \alpha)] = -\omega A\ \sin(\omega t + \alpha) = Re\left(\frac{\mathrm{d}z(t)}{\mathrm{d}t}\right) \\
\frac{\mathrm{d}z(t)}{\mathrm{d}t} &= i\omega A e^{i(\omega t + \alpha)} = i\omega z = -\omega A\ \sin(\omega t + \alpha) + i\omega A\ \cos(\omega t + \alpha)
\end{aligned}
\tag{9}
$$

c) acceleration

$$a(t) = \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = -\omega^2 A \ \cos(\omega t + \alpha) = Re\left(\frac{\mathrm{d}^2 z(t)}{\mathrm{d}t^2}\right)$$
$$\frac{\mathrm{d}^2 z(t)}{\mathrm{d}t^2} = (i\omega)^2 A e^{i(\omega t + \alpha)} = -\omega^2 z \tag{10}$$



II. THE SUPERPOSITION OF PERIODIC MOTIONS

Superposed vibrations in one dimension:
The resultant of two or more vibrations is the sum of the individual vibrations.

Remarks:
Is the displacement produced by two disturbances, acting together, equal to the superposition of the displacements as they would occur separately?

Yes or no depending whether or not the displacements is strictly proportional to the force producing it, i.e. $F = m\frac{d^2x}{dt^2} = -kx$.

If simple addition holds, **the system is** said to be **linear**.

The superposed vibrations of equal frequency

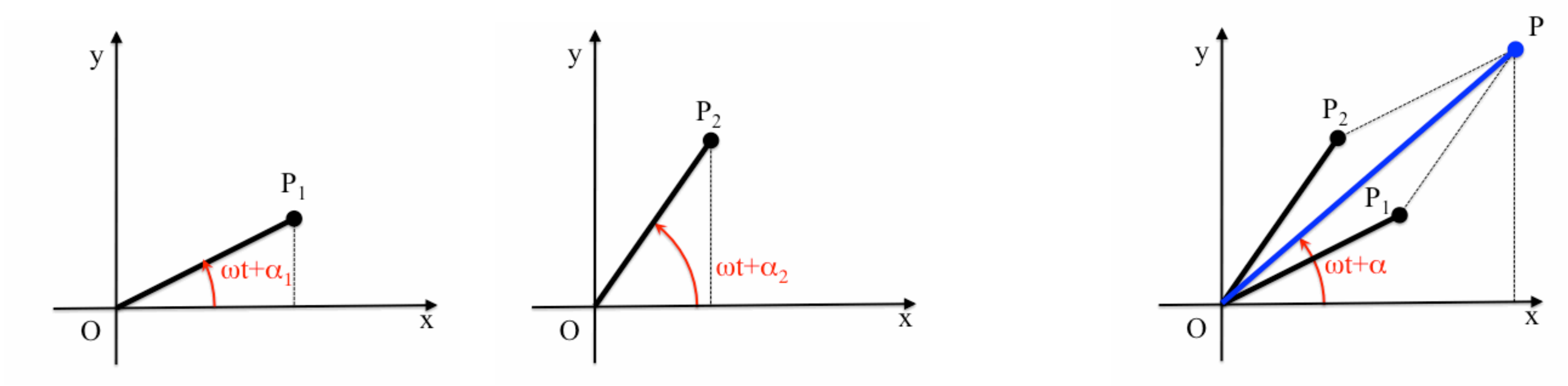
Problem:

We have two SHMs:

$$
\begin{aligned}
x_1(t) &= A_1 \cos(\omega t + \alpha_1) \\
x_2(t) &= A_2 \cos(\omega t + \alpha_2)
\end{aligned}
$$

and we wish to express their superposition as a single SHM:

$$
\begin{aligned}
x = x_1 + x_2 &= A_1 \cos(\omega t + \alpha_1) + A_2 \cos(\omega t + \alpha_2) \\
&= A \cos(\omega t + \alpha)
\end{aligned}
$$



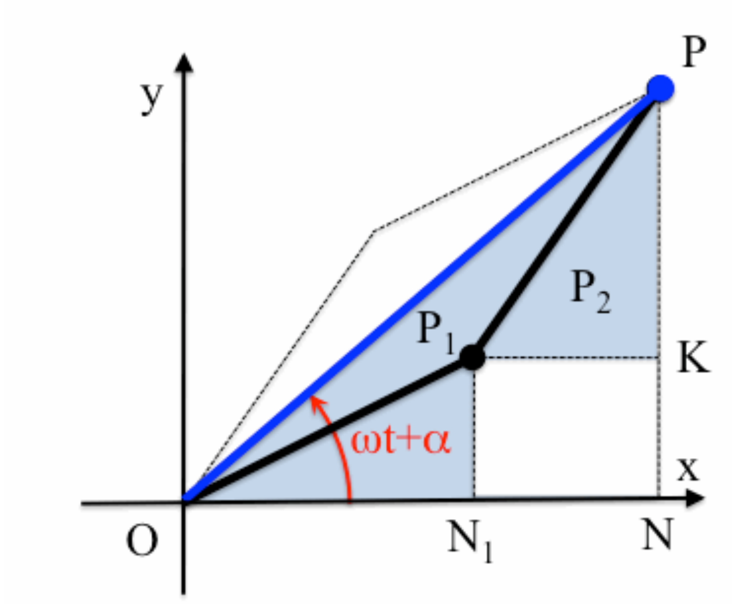
Solution can be obtained in geometric terms using rotating vector representation of SHM:

OP_1 is a rotating vector of length A_1 making an angle $(\omega t + \alpha_1)$ with the axis x at t
OP_2 is a rotating vector of length A_2 making an angle $(\omega t + \alpha_2)$.

The sum of these is the vector OP defined by the parallelogram law of vector addition:

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\alpha_2 - \alpha_1) \tag{1}$$

where $\alpha_2 - \alpha_1$ is the angle between OP_1 and P_1P.



Derivation:

$ON: \quad A_1 \ \cos(\omega t + \alpha_1) + A_2 \ \cos(\omega t + \alpha_2)$

$NP: \quad A_1 \ \sin(\omega t + \alpha_1) + A_2 \ \sin(\omega t + \alpha_2)$

$$\begin{aligned} A^2 &= A_1^2 \ \sin^2(\omega t + \alpha_1) + A_2^2 \ \sin^2(\omega t + \alpha_2) + 2A_1A_2 \ \sin(\omega t + \alpha_1)\sin(\omega t + \alpha_2) + \\ &+ A_1^2 \ \cos^2(\omega t + \alpha_1) + A_2^2 \ \cos^2(\omega t + \alpha_2) + 2A_1A_2 \ \cos(\omega t + \alpha_1)\cos(\omega t + \alpha_2) \\ &= A_1^2 + A_2^2 + 2A_1A_2\cos(\alpha_2 - \alpha_1) \end{aligned}$$

where we used the trigonometric identities

$$\begin{aligned} \sin^2\phi + \cos^2\phi &= 1 \\ \sin\phi \ \sin\psi &= \frac{1}{2}[\cos(\phi - \psi) - \cos(\phi + \psi)] \\ \cos\phi \ \cos\psi &= \frac{1}{2}[\cos(\phi - \psi) + \cos(\phi + \psi)] \end{aligned}$$

Use of complex exponential formalism:

the rotating vectors:

$$OP_1: \quad z_1 = A_1\, e^{i(\omega t+\alpha_1)}$$
$$OP_2: \quad z_2 = A_2\, e^{i(\omega t+\alpha_2)}$$

the resultant:

$$\begin{aligned} z = z_1 + z_2 &= A_1 e^{i(\omega t+\alpha_1)} + A_2\, e^{i(\omega t+\alpha_2)} \\ &= e^{i(\omega t+\alpha_1)} \left[A_1 + A_2\, e^{i(\alpha_2-\alpha_1)}\right] \end{aligned}$$

- a vector of length A_2 is to be added at an angle $(\alpha_2 - \alpha_1)$ to a vector of length A_1;
- the factor $e^{i(\omega t+\alpha_1)}$ tells us that the whole diagram rotates by the angle $(\omega t + \alpha_1)$.

From the complex formalism, we get the amplitude of the resultant motion

$$
\begin{aligned}
A^2 &= |z|^2 = z\, z^* \\
&= \left\{ e^{i(\omega t+\alpha_1)} \left[A_1 + A_2\, e^{i(\alpha_2-\alpha_1)} \right] \right\} \left\{ e^{-i(\omega t+\alpha_1)} \left[A_1 + A_2\, e^{-i(\alpha_2-\alpha_1)} \right] \right\} \\
&= A_1^2 + A_2^2 + 2A_1 A_2 \cos(\alpha_2 - \alpha_1)
\end{aligned}
$$

Superposed vibrations of the same frequency and amplitude

In general, the values of A and α cannot be further simplified, apart from special cases.

Special case $A_1 = A_2$:

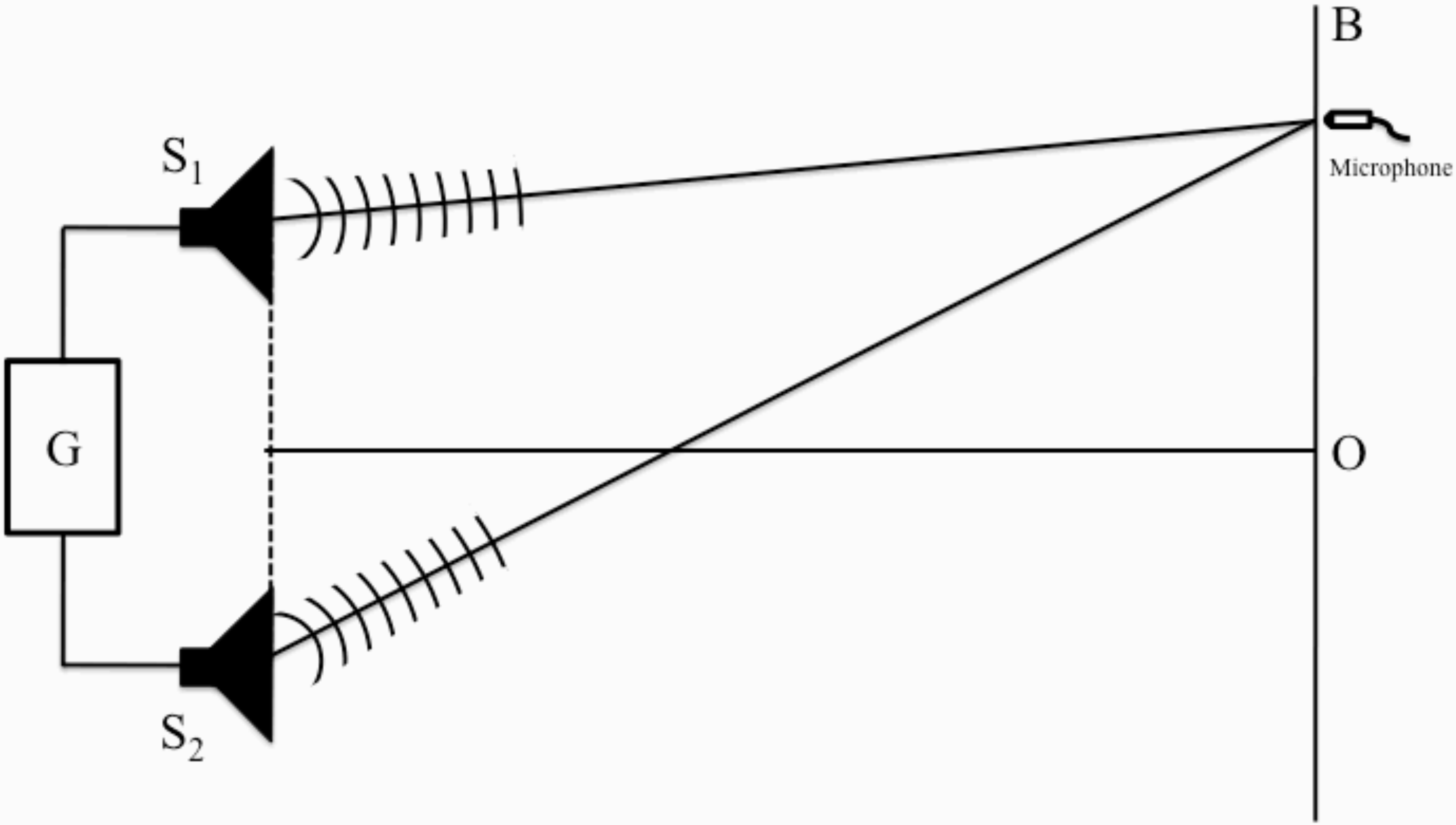
$$
\begin{aligned}
A &= \sqrt{A_1^2 + A_1^2 + 2A_1A_1\cos(\alpha_2 - \alpha_1)} \\
&= 2A_1\sqrt{\frac{1 + \cos(\alpha_2 - \alpha_1)}{2}} \\
&= 2A_1\cos\left(\frac{\alpha_2 - \alpha_1}{2}\right)
\end{aligned}
$$

The situation when

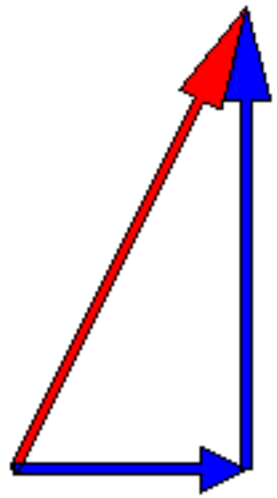
$$A = 2A_1 \ \cos\left(\frac{\alpha_2 - \alpha_1}{2}\right) = 2A_1 \ \cos\delta$$

occurs for example if two identical loudspeakers are driven sinusoidally from the same signal generator and sound waves are picked up by a microphone at a fairly distant point.

If the microphone is moved along the line OB, the phase difference $\delta = \alpha_2 - \alpha_1$ increases steadily from the initial value zero at the point O. If the wavelength of the sound waves is shorter than separation of the speakers, the resultant amplitude A can be observed changing between zero and the maximum $2A_1$.


B
Microphone
S_1
G
O
S_2

We know that oscillations in one dimension can be viewed as the projection of a rotating vector in two dimensions. This means that the superposition of two oscillations can be represented as the addition of two time-dependent complex vectors. The below animation shows this for two oscillations of the same frequency but different amplitudes (one is twice as big as the other) and phases (a difference of):



Note that the sum -- the red arrow -- does not change its length, so the amplitude of the superposition is constant over time.

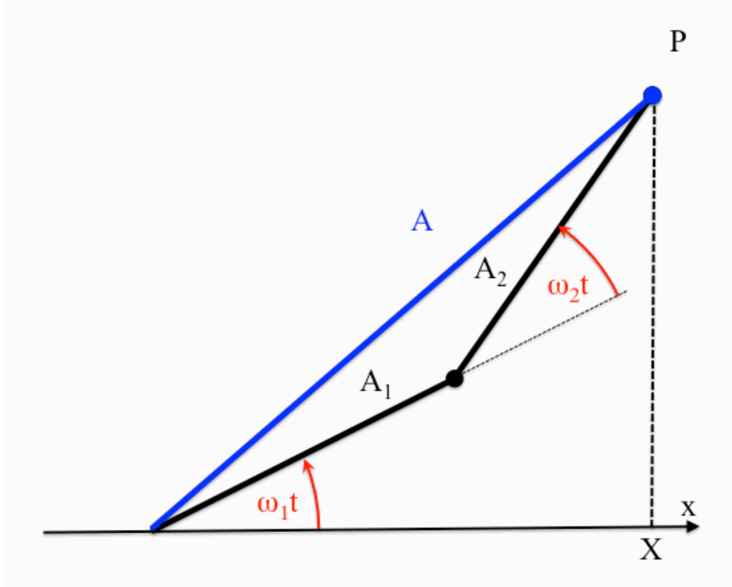
Superposed vibrations of different frequency. Beats

We consider two vibrational motions characterized by the displacements

$$x_1 = A_1 \cos(\omega_1 t + \alpha_1)$$
$$x_2 = A_2 \cos(\omega_2 t + \alpha_2)$$

Now not only $A_1 \neq A_2$ but also $\omega_1 \neq \omega_2$ and thus

the phase difference between the vibrations is now continually changing.

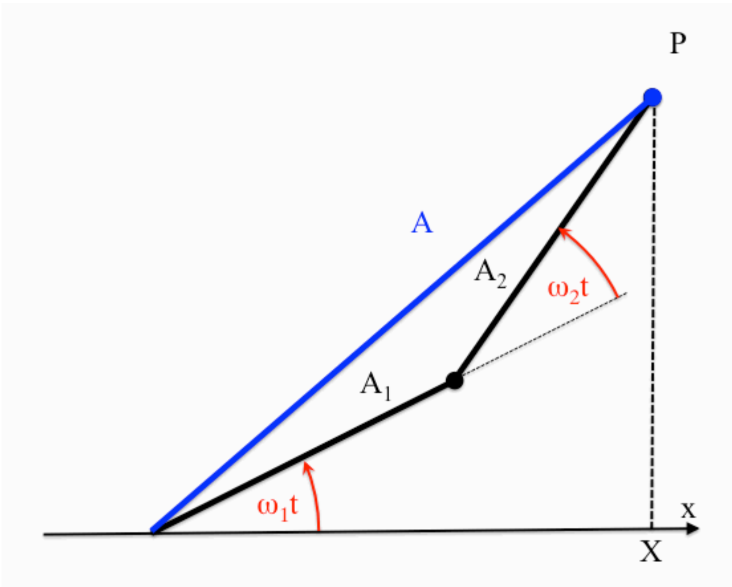


Since the initial phase difference $\alpha_2 - \alpha_1$ is not now of major significance, we take $\alpha_1 = \alpha_2 = 0$ for the sake of simplicity:

$$\begin{aligned} x_1 &= A_1 \cos(\omega_1 t) \\ x_2 &= A_2 \cos(\omega_2 t) \end{aligned}$$

At some arbitrary instant, the combined displacements correspond to the lenght OX.

The length OP of the combined vectors must lie between the sum and difference of A_1 and A_2; the magnitude of the displacement OX may be between zero and $A_1 + A_2$.



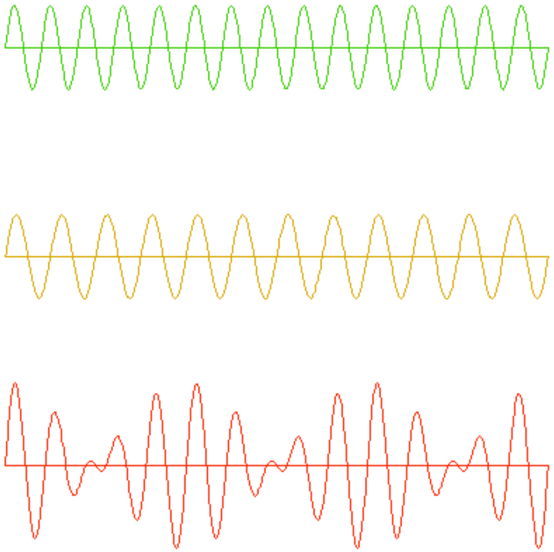
Unless there is some simple relation between ω_1 and ω_2, the resultant displacement will be a complicated function of time, perhaps even repeating itself.

The condition for any sort of true periodicity in the combined motion is that the periods of the component motions be commensurate, i.e. there exist integers n_1 and n_2 such that

$$T = n_1 T_1 = n_2 T_2 \tag{1}$$

where T is the period of the combined motion for the smallest integral values of n_1 and n_2.

The general appearance of the combined motion is not particularly simple even for the case where the component frequencies are commensurate with small integers n_1 and n_2. Also the resultant may strongly depend on the relative initial phase of the combining vibrations in the case of vibrations of incommensurable periods.



Beats : If two SHMs are close in frequency,

$$\begin{aligned} x_1 &= A \cos(\omega_1 t) \\ x_2 &= A \cos(\omega_2 t) \end{aligned}$$

the combined disturbance exhibits **beats**:

$$x = 2A\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t\right) \tag{2}$$

where the first cosine term is slowly oscillating and the second is fast oscillating.

The condition for beats:

$$|\omega_1 - \omega_2| << \omega_1 + \omega_2 \tag{3}$$

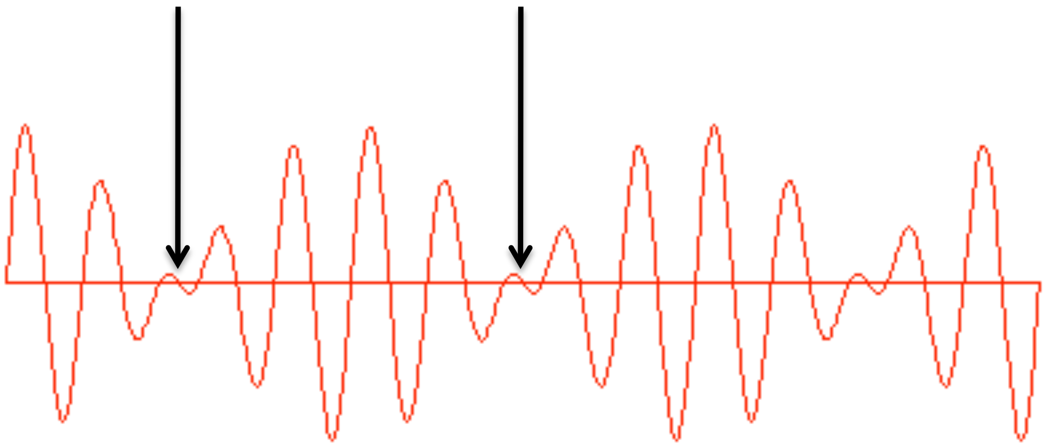
The combined displacement can be fit within an envelope defined by the pair of equations

$$x = \pm 2A \cos\left(\frac{\omega_1 - \omega_2}{2} t\right) \tag{4}$$

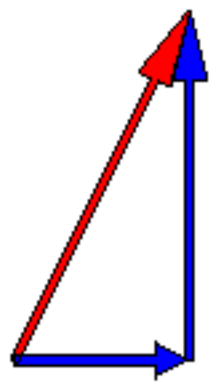
which describes a relatively slow amplitude modulation of the combined oscillations.

The time between successive zeros of the modulating disturbance is one half-period of the modulating factor as described in equation above, i.e.

$$T_{beat} = \frac{2\pi}{|\omega_1 - \omega_2|} \tag{5}$$



The next animation illustrates what happens when the frequencies are different, with one being 3π rad.s^{-1} and the other being 4π rad.s^{-1} :



In this case, the length of the sum *does vary*, so the superposition of the two oscillations produces a time-dependent amplitude.

This phenomenon is called "beating", and is most pronounced when the two frequencies are close together. The below animation shows two oscillations of frequencies $\omega_1 = 6\pi$ rad.s^{-1} and $\omega_2 = 0.8\ \omega_1$, followed by the sum of the two. The result is a "sinewave inside a sinewave". This is the principle behind the amplitude modulation that's used for AM radio.