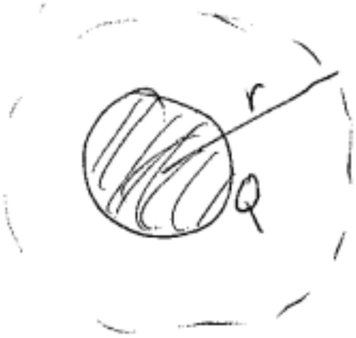
Application 3b

but spherically symmetric (isotropic)

Sphere carrying charge (uniform or hollow)

Doesn't matter outside.

Matters inside



Outside

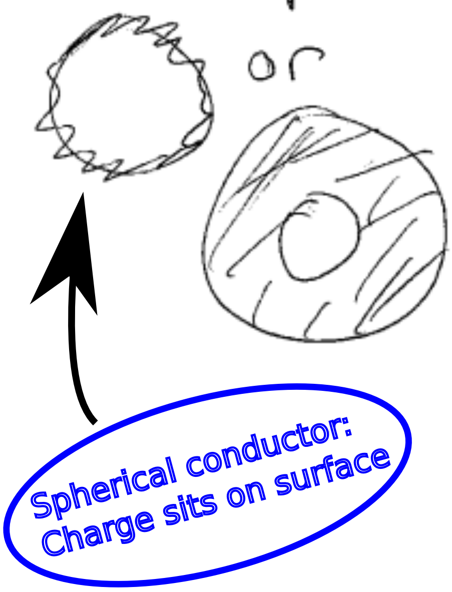
$$\oint \vec{E}\cdot d\vec{S} = E\cdot 4\pi r^2 = \frac{Q}{\epsilon_0}$$

$$\Rightarrow E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$$

as if it was a pt. charge

Applic. 3c

Spherical conductor or hollow charged sphere



Inside

$$E\cdot 4\pi r^2 = \frac{0}{\epsilon_0} = 0$$

$$\Rightarrow E = 0$$

Actually, $E = 0$ inside ANY conductor

Applicat^n 3d

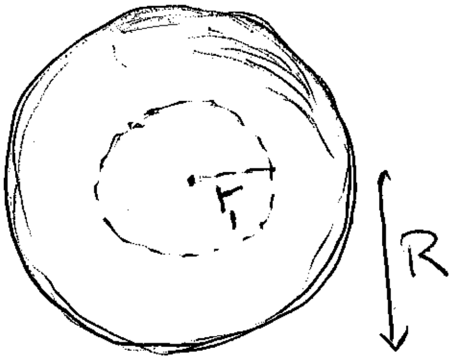
Inside, charged sphere, density $\rho(r)$

$$E\cdot 4\pi r^2 = \frac{1}{\epsilon_0}\int_0^r dr_1\, 4\pi r_1^2\, \rho(r_1)$$

Application 3d

Inside charged sphere, density $\rho(r)$

[$\rho(r)$ depends only on r, not on θ, φ, spherically symmetric]



Use spherical Gaussian surface, radius $r_1 < R$, concentric with physical sphere.

Apply Gauss' law:

$$E \cdot 4\pi r_1^2 = \frac{1}{\epsilon_0}\,(\text{charge enclosed})$$

$$= \frac{1}{\epsilon_0} \int_{r<r_1} dV\, \rho(r)$$

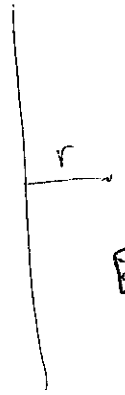
$$= \frac{1}{\epsilon_0} \int_0^{r_1} dr\, 4\pi r^2\, \rho(r)$$

$$\Rightarrow E = \frac{1}{\epsilon_0 r_1^2} \int_0^{r_1} dr\, r^2 \rho(r) \quad \ldots\ldots \text{inside}$$

Outside? $E = \frac{1}{4\pi\epsilon_0} Q_{total}$

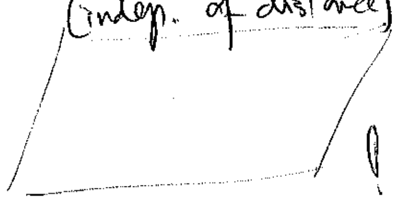
* Electric field due to infinitely extended charged objects

• $E \propto \frac{1}{r^2}$



$E \propto \frac{1}{r}$

$E \sim$ constant (indep. of distance)



→ Infinite charged objects are idealized situations (limiting cases), but useful to study

↙ possible to derive simple expressions

↘ sometimes serve as good approximations to realistic situations.

* Electric potential : problems with infinite geometries

$$V(\vec{r}) = V(\vec{r}_0) - \int_{\vec{r}_0}^{\vec{r}} \vec{E} \cdot \vec{dl}$$

Usually, reference chosen as $\vec{r}_0 = \infty$ and $V(\infty) \equiv 0$.

However, for infinite line/plane, $V(\infty) = \infty$!
→ have to choose different reference

* Electric charge (density), potential, field

$\vec{E}$ from ρ : $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3\vec{r}\,' \rho(\vec{r}\,') \frac{\widehat{\vec{r}-\vec{r}\,'}}{|\vec{r}-\vec{r}\,'|^2}$

ρ from $\vec{E}$: $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

V from ρ : $V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3\vec{r}\,' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$

ρ from V : $\nabla^2 V = -\frac{\rho}{\epsilon_0}$

$\vec{E}$ from V : $\vec{E} = -\vec{\nabla} V$ (will need correction for dynamics)

V from $\vec{E}$: $V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}(\vec{r}\,') \cdot d\vec{r}\,'$

with $\vec{r}_0$ usually at ∞

* Potential defined only UP TO A CONSTANT

Ex. $V_1(\vec{r}) = \alpha yz - \beta x$ and
$V_2(\vec{r}) = \alpha yz - \beta x + \gamma$ represent SAME ~~PHYSICAL~~ Electrostatic SYSTEM

* We've been studying ELECTROSTATICS
  → charges at rest.

charges at rest — generate $\vec{E}$-fields
— are affected by $\vec{E}$-field (experiences a force)

charges in <u>motion</u> have additional physics
e.g., — generate $\vec{B}$-fields
— ~~generate~~ are affected by $\vec{B}$-fields (experience forces)

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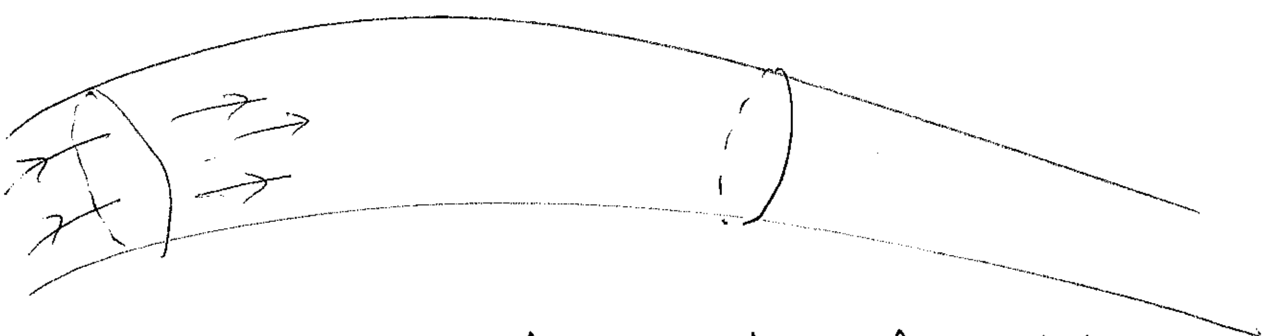
* <u>Electric Currents</u>

- Usually consider currents through a WIRE (possible to generalize to other situations)



Usually carried by charged particles, often electrons, each with charge (−)e.

* Electric current through a surface S

Defined as the charge Q passing thru S per unit time

$$I = \frac{dQ}{dt} \quad (\text{a SCALAR})$$

Units: Amperes = Coulomb/sec

For a wire, S would be taken as a cross-section of the wire.

* Current Density $\vec{J}$

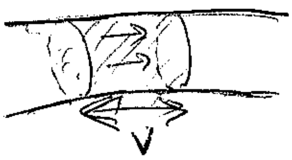
Defined as a vector field whose

- direction is the velocity vector of the conduction carriers (electrons/holes)
- magnitude is the amount of charge crossing a unit ~~of~~ perpendicular area per unit time

* If n is the density of carriers and q is the charge of each carrier and $\vec{v}$ is the av. velocity of carrier,

then $\boxed{\vec{J} = q n \vec{v}} = \rho \vec{v}$

$q = -e$ for electrons

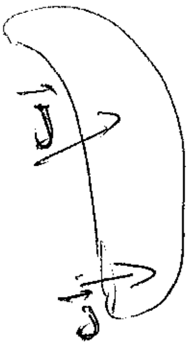


In unit time, the carriers in this volume would cross the area: $v.An$ carriers, with charge $qvAn$

$$\Rightarrow \quad j = \frac{qvAn}{A} = qnv$$

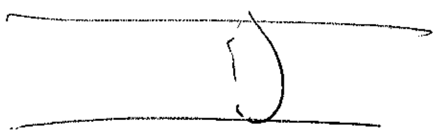
---

* Comparing definitions of current I through S and of current density $\vec{J}$:



$$I = \int \vec{J} \cdot d\vec{S}$$

If $\vec{J}$ is uniform in wire

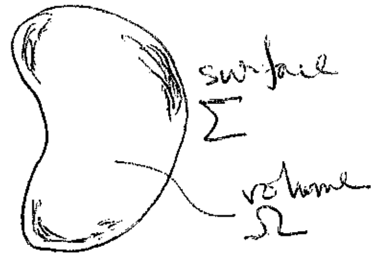


$$I = JA = qnvA$$

* In real material, $\vec{v}$ is an "average" or "drift" velocity.

The CONTINUITY EQUATION

Consider closed surface Σ,
enclosing region Ω



Current outward thru Σ:

$$I = \oint_{\Sigma} \vec{J} \cdot \vec{dS} = \int_{\Omega} \left(\vec{\nabla} \cdot \vec{J}\right) dV$$

But also: $$I = -\frac{d}{dt} Q_{enc.} = -\frac{d}{dt} \int_{\Omega} \rho \, dV = -\int_{\Omega} \left(\frac{\partial \rho}{\partial t}\right) dV$$

Thus $$\int_{\Omega} \left(\vec{\nabla} \cdot \vec{J}\right) dV = \int_{\Omega} \left(-\frac{\partial \rho}{\partial t}\right) dV$$

This is valid for <u>ANY</u> region/volume Ω

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}} \qquad \boxed{\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0}$$

CONTINUITY EQUATION

Mathematical statement of local charge conservation

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Ohm's law

microscopic form

$$\vec{J} = \sigma \vec{E}$$

$$\vec{E} = \rho \vec{J}$$

macroscopic form

$$V = R I$$

not a fundamental law — holds for "most" materials approximately.

$\rho = 1/\sigma$ is material-dependent.

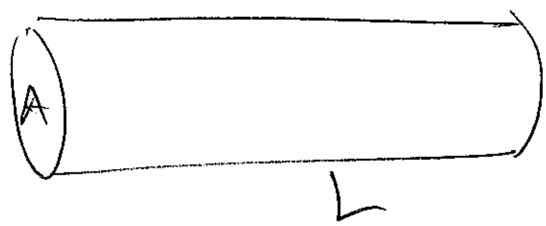
R is material + shape-dependent

$\rho = \frac{1}{\sigma}$ strongly TEMPERATURE-dependent

σ = conductivity, ρ = resistivity
R = resistance

For straight wire

cross-section A
length L



$$V = |\vec{E}| \cdot L \quad , \quad I = J \cdot A$$

$$|\vec{E}| = \rho |\vec{J}| \Rightarrow V = \left(\frac{\rho L}{A}\right) I$$

$$\Rightarrow R = \frac{\rho L}{A} = \frac{L}{\sigma A}$$

better written as ΔV, although V is common. Difference betn two ends.

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* Moving charges — many different situations possible.

steady currents,

point charges in motion,

a wire carrying current, itself in motion, etc.

---

* MAGNETIC FIELD

A moving charge is affected by a $\vec{B}$-field, according to the Lorentz force law:

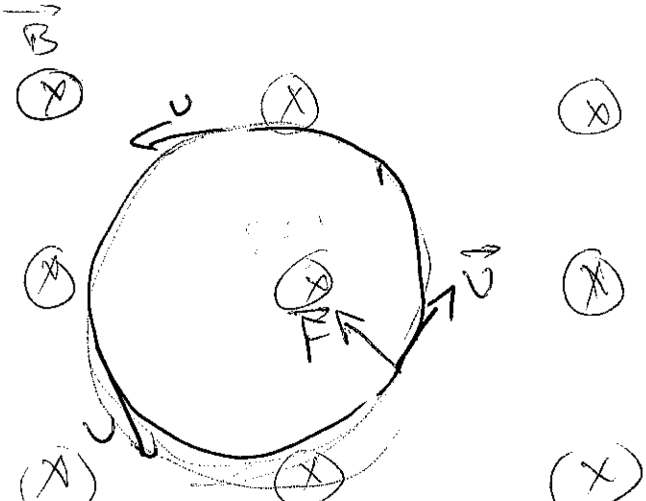
A charge q with velocity $\vec{v}$ in a magnetic field $\vec{B}$ experiences the force

$$\boxed{\vec{F} = q\vec{v} \times \vec{B}}$$

If subject to both electric field $\vec{E}$ and magnetic field $\vec{B}$, then,

$$\boxed{\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}}$$

* Cyclotron motion



Imagine $\vec{v}$ perpendicular to a uniform $\vec{B}$. The charged particle will experience force <u>perpendicular</u> to it's velocity. This will act as a centripetal force, and the trajectory is circular.

$$qvB = \frac{mv^2}{R}$$

Lorentz force — centripetal force

$$\Rightarrow \boxed{R = \frac{mv}{qB}}$$

radius of cyclotron motion

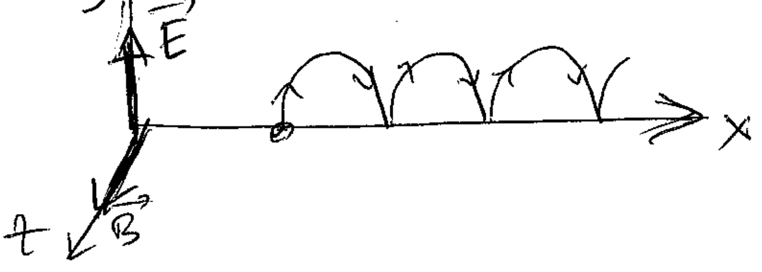
cyclotron frequency

$$\frac{\omega}{2\pi} = \frac{v}{2\pi R}$$

$$\Rightarrow \omega = \frac{qB}{m}$$

actually an angular freq.

* Combined $\vec{E}$, $\vec{B}$ fields can lead to various types of trajectories. Eg., particle starting at rest



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* Magnetic forces do no work,

When a charged particle moves by $d\vec{l} = \vec{v}\,dt$,

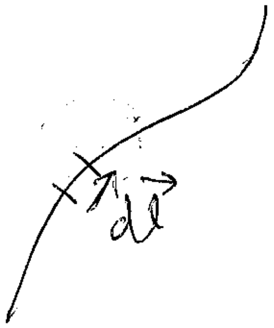
The work done by the magnetic field is

$$dW_{mag} = \vec{F}_{magn} \cdot d\vec{l} = (q\vec{v} \times \vec{B}) \cdot \vec{v}\,dt$$

$$= 0$$

because $\vec{v} \times \vec{B}$ is perpendicular to $\vec{v}$.

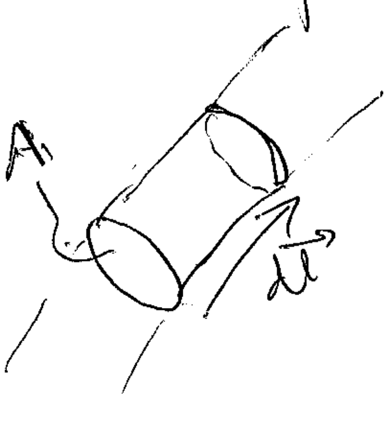
Magnetic force on wire carrying steady current



Force on element $\vec{dl}$

$$d\vec{F}_{mag} = I\left(\vec{dl} \times \vec{B}\right)$$

Justification : $I\vec{dl}$ takes the role of $q\vec{v}$



$$q \longrightarrow \rho A |\vec{dl}|$$

$$q\vec{v} \longrightarrow \rho A |\vec{dl}| \vec{v} = \rho v A \vec{dl}$$

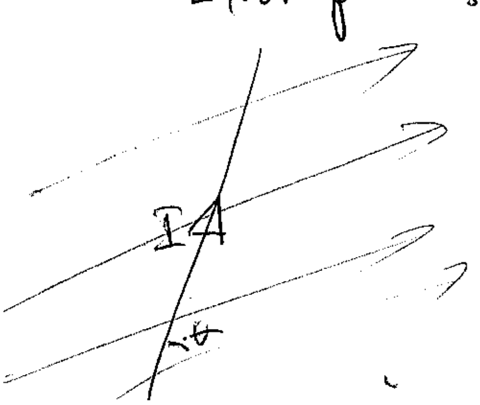
$$= J A \vec{dl} = I\vec{dl}$$

Thus $q\vec{v} \times \vec{B}$ is replaced by $I\left(\vec{dl} \times \vec{B}\right)$

for an infinitesimal element

Full wire : $\vec{F}_{mag} = I \int \vec{dl} \times \vec{B}$

Example : long straight wire in constant field :



$$F = I \int dl \, B \sin\theta = IB\sin\theta \int dl$$

Force on segment of length $L = IBL\sin\theta$

Force per UNIT LENGTH $= IB\sin\theta$

MAXWELL'S SECOND EQUATION

* Analogy to Gauss' law would be

$$\oint \vec{B} \cdot \vec{dS} = \frac{Q_M}{\text{constant}}$$

where Q_M is a "magnetic charge"

* Experimental fact: there exists <u>no</u> magnetic charge or magnetic monopole — magnetic poles come in pairs.

* Thus $\boxed{\oint_\Sigma \vec{B} \cdot \vec{dS} = 0}$ for ANY closed surface Σ

* Using Gauss' divergence theorem,

$$\oint_\Sigma \vec{B} \cdot \vec{dS} = \int_{\substack{\text{enclosed} \\ \text{region}}} \left(\vec{\nabla} \cdot \vec{B} \right) dV = 0$$

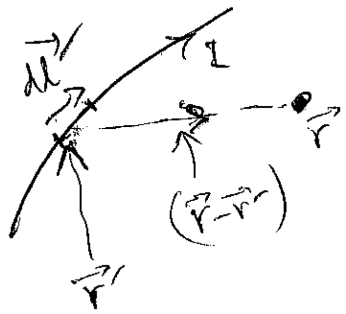
for ANY region

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

Maxwell's second equation

* The magnetic field produced by a steady current

Biot-Savart law



Magnetic field at point $\vec{r}$ due to the element $\vec{dl}'$ at position $\vec{r}'$

$$\vec{dB} = \frac{\mu_0 I}{4\pi} \frac{\vec{dl}' \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} = \frac{\mu_0 I}{4\pi} \frac{\vec{dl}' \times (\widehat{r-r'})}{|\vec{r}-\vec{r}'|^2}$$

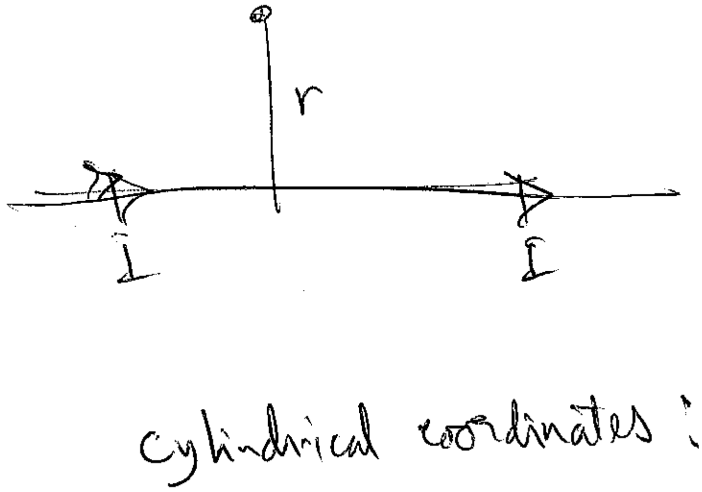
μ_0 = permeability of free space

$= 4\pi \times 10^{-7}$ N/A^2

Unit of $\vec{B}$: Tesla

1 Tesla = 1 N/A.m = 10^4 gauss

* Long infinite wire



$$B = \frac{\mu_0 I}{2\pi r}$$

r is the CYLINDRICAL DISTANCE

cylindrical coordinates : $\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{e}_\phi$

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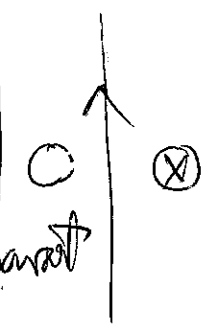
Long infinite wire

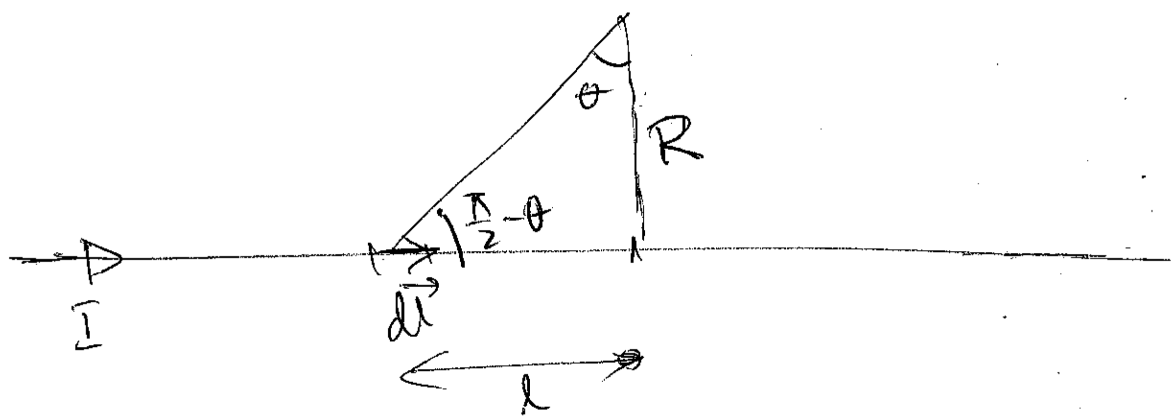
Right-hand-rule for field due to current

Follows from Biot-Savart rule



Derivation of $B = \frac{\mu_0 I}{2\pi (\text{distance})}$

using Biot-Savart



$$\vec{dB} = \frac{\mu_0 I}{4\pi} \frac{\vec{dl} \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^2}$$

$$= \frac{\mu_0 I}{4\pi} \frac{dl \cdot 1 \cdot \sin\left(\frac{\pi}{2}-\theta\right)}{l^2+R^2} \hat{n}$$

$\hat{n}$ out of paper

$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta = \frac{R}{\sqrt{l^2+R^2}}$

$$dB = \frac{\mu_0 I R}{4\pi} \frac{dl}{(l^2+R^2)^{3/2}}$$

$\vec{dB}$ from each element points in the same direction $\hat{n}$. Scalar integration sufficient.

$$B = \int dB = \frac{\mu_0 I R}{4\pi} \int \frac{dl}{(l^2+R^2)^{3/2}}$$

Limits? For segment of Length L, integrate from $l = -\frac{L}{2}$ to $l = +L/2$.

For infinite wire, integrate from $l = -\infty$ to $l = +\infty$ (or take limit $L \to \infty$).

I know that $\frac{d}{du}\left(\frac{u}{(u^2+a^2)^{1/2}}\right) = \frac{a^2}{(u^2+a^2)^{3/2}}$

or $$\int \frac{du}{(u^2+a^2)^{3/2}} = \frac{u}{a^2\sqrt{u^2+a^2}}$$

Thus B from finite segment

$$B = \frac{\mu_0 I R}{4\pi}\int_{-L/2}^{L/2} \frac{dl}{(l^2+R^2)^{3/2}} = \frac{\mu_0 I R}{4\pi}\cdot\frac{1}{R^2}\frac{l}{\sqrt{l^2+R^2}}\Bigg]_{-L/2}^{L/2}$$

$$B = \frac{\mu_0 I}{4\pi R}\frac{L}{\sqrt{\left(\frac{L}{2}\right)^2+R^2}}$$

Infinite wire: take limit $L \to \infty$

$$B = \frac{\mu_0 I}{4\pi R} \cdot 2 = \frac{\mu_0 I}{2\pi R}$$

Or: integrate directly from $l = -\infty$ to $l = +\infty$

(Exercise)

<u>Alternative!</u> Work with angle θ

$$dB = \frac{\mu_0 I R}{4\pi} \frac{dl}{(l^2+R^2)^{3/2}}$$

$$= \frac{\mu_0 I R}{4\pi} \frac{R \sec^2\theta \, d\theta}{R^3 \sec^3\theta}$$

$$= \frac{\mu_0 I}{4\pi R} \cos\theta \, d\theta$$

$$\left\{ \begin{aligned} l &= R \tan\theta \\ dl &= R\left[\frac{d}{d\theta}\tan\theta \, d\theta\right] \\ &= R \sec^2\theta \, d\theta \\ (l^2+R^2) &= R^2(1+\tan^2\theta) \\ &= R^2 \sec^2\theta \end{aligned} \right.$$

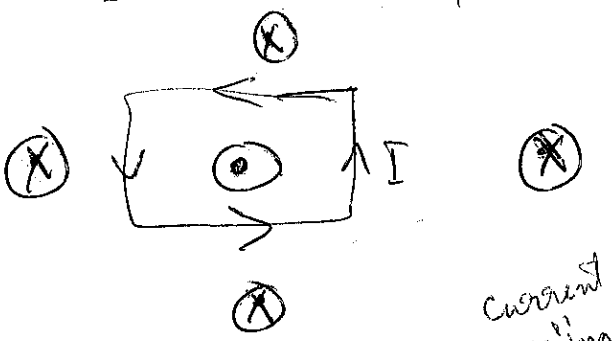
Infinite wire:

limits $\theta = -\frac{\pi}{2}$ to $\theta = +\frac{\pi}{2}$

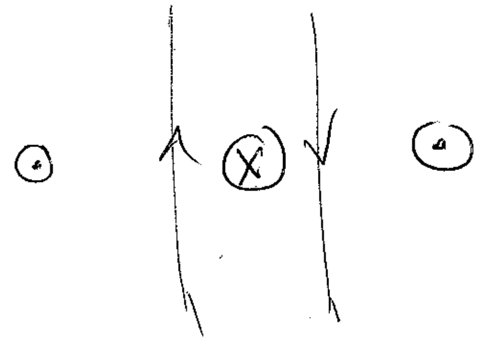
$$B = \frac{\mu_0 I}{4\pi R} \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta = \frac{\mu_0 I}{4\pi R}\left[\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right]$$

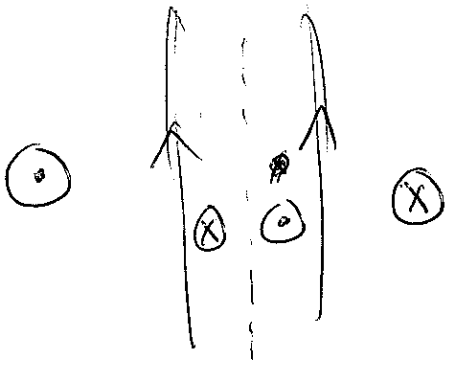
$$= \mu_0 I / 2\pi R$$

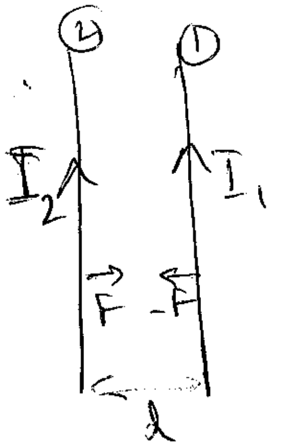
* Directions of magnetic field due to current combinations



current loop ≈ "magnet"







Two infinite wires.

Opposite currents repel, parallel currents attract.

Can see this using $\vec{F} = I(d\vec{l} \times \vec{B})$ — (α)

and $d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^2}$ — or $\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$

or $B = \frac{\mu_0 I}{2\pi d}$

Current (1) produces, at position of current (2), $\vec{B}$-field pointing OUTWARD.

Cross-product of Eq. (α) then gives ATTRACTION.

Strength of attracti / repulsive force?

Force on infinite wire (2) is INFINITE.

Force on segment of length L

$$= I_2 L B \sin\left(\frac{\pi}{2}\right) = I_2 L \cdot \frac{\mu_0 I_1}{2\pi d}$$

$$= \frac{\mu_0 I_1 I_2}{2\pi d} L$$

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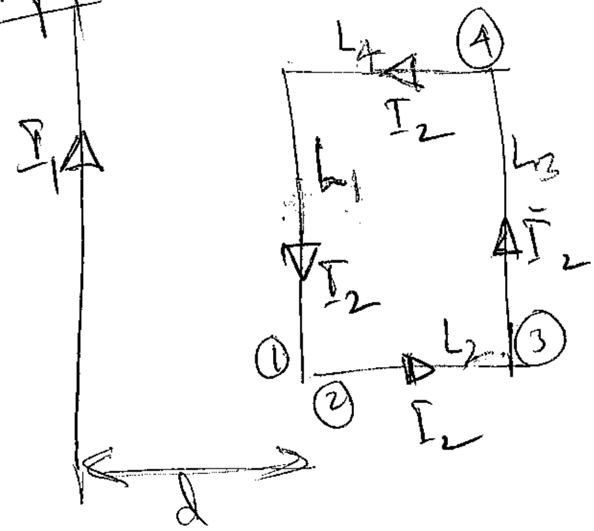
Force on unit length $= \dfrac{\mu_0 I_1 I_2}{2\pi d}$

Example



Push on ① $= \dfrac{\mu_0 I_1 I_2}{2\pi d} L_1$

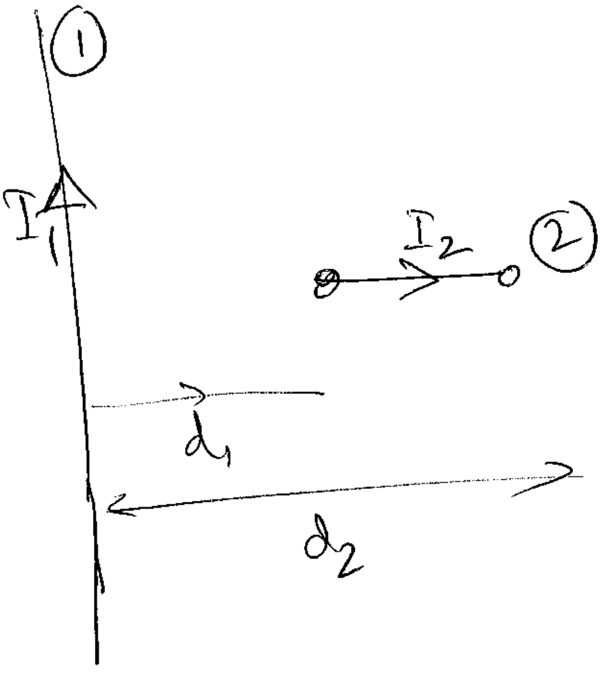
Pull on ③ $= \dfrac{\mu_0 I_1 I_2}{2\pi (d+L_2)} L_3$

Sideways forces on ② & ④.

Not so easy to calculate. b/c magnetic field varies along these segments.

Can calculate by integration.

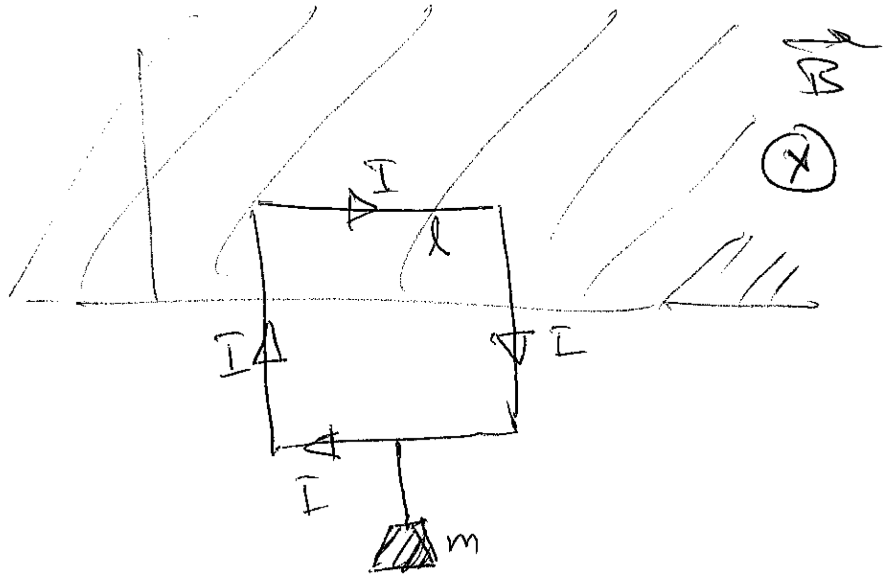
Exercise



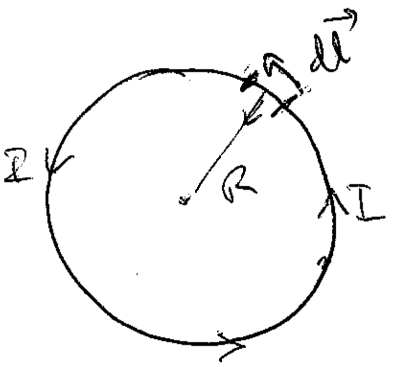
Calculate force on segment ② due to current thru long wire ①

Another example
(Griffiths)

$$I l B = mg$$



* CIRCULAR RING carrying current



$$\vec{dB} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times (\widehat{\vec{r}-\vec{r}'})}{|\vec{r}-\vec{r}'|^2}$$

$$= \frac{\mu_0 I}{4\pi} \frac{dl \cdot 1 \cdot \sin\frac{\pi}{2}}{R^2} \hat{n}$$

$$dB = \frac{\mu_0 I}{4\pi R^2} dl$$

Each element contributes the same!

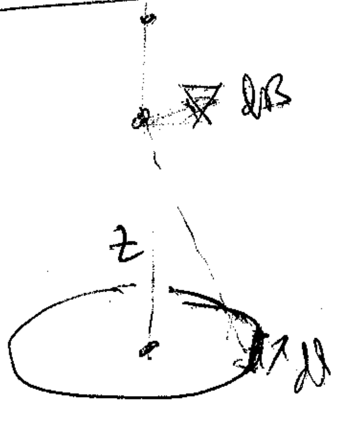
$$B = \int dB = \frac{\mu_0 I}{4\pi R^2} \int dl = \frac{\mu_0 I}{4\pi R^2} (2\pi R)$$

No need to parametrize position of element.

B (at center of circular loop)

$$\boxed{B = \frac{\mu_0 I}{2R}} \quad \text{--- } (\beta)$$

On axis, but away from the plane of the loop!



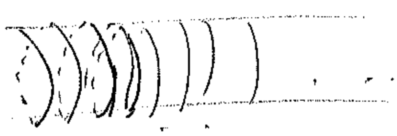
All elements still contribute same magnitude, but different directions. Need to take component:

$$B = \frac{\mu_0 I}{2} \frac{R^2}{(R^2+z^2)^{3/2}} \quad \text{--- } (\alpha)$$

Exercise 1: Derive Eq. (α). (Difficult)

Exercise 2: Show that in-plane formula (β) is recovered for $z=0$.

SOLENOID

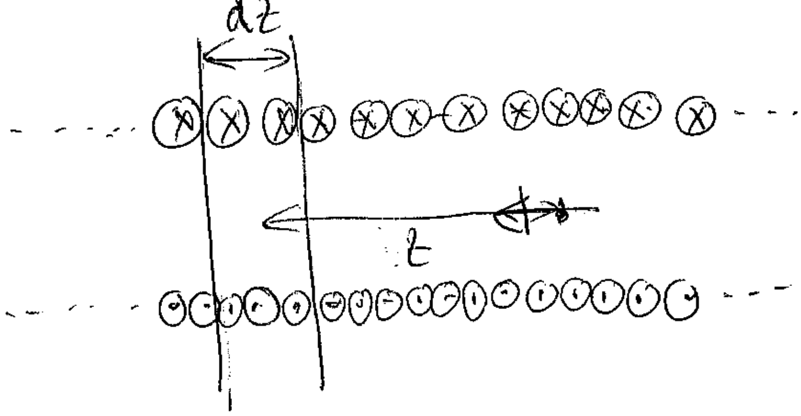


(Helical coil, tightly packed)

Used to produce strong, nearly uniform $\vec{B}$-field

Calculating field on the axis:

Radius R, current I, n turns per unit length.



Consider ring-like element of solenoid

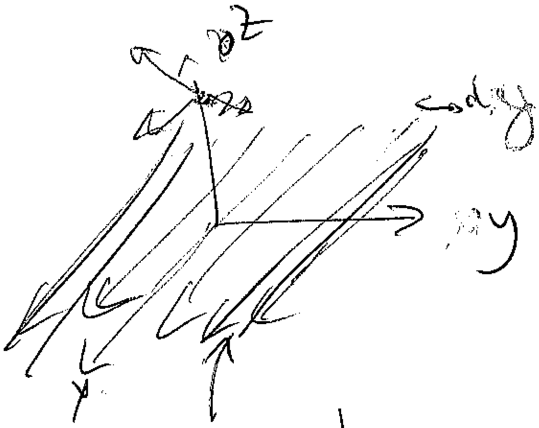
$$dB = \frac{\mu_0 (dI)}{2} \frac{R^2}{(R^2+z^2)^{3/2}}$$

with $dI = nI\,dz$

Exercise Integrate over z (for $z = -\frac{L}{2}$ to $z = +\frac{L}{2}$ then $L \to \infty$)

Answer: for infinite solenoid: $\boxed{B = \mu_0 n I}$

* Thin sheet of current. optional



current thin dy $= K dy$

Curr. on xy-plane with surface current density $\vec{K} = K \hat{i}$

$\vec{B}$-field in $-\hat{j}$ direction for $z > 0$ and in $+\hat{j}$ direction for $z < 0$.

Exercise

Write dB_y (dB_z components cancel)

Integrate to get $\vec{B} = \frac{\mu_0}{2} K (\mp \hat{j})$