8

* ~~Unit of potential : Joule/Coulomb = Volt~~

~~Unit of $\vec{E}$-field : N/Coulomb = Volt-m~~

* <u>Fundamental properties of $\vec{E}$-field</u>

A) Curl

(A1) $\vec{\nabla}\times\vec{E} = 0$ (A2) $\oint \vec{E}\cdot d\vec{l} = 0$ for any closed path — often $\vec{E}\cdot d\vec{r}$

line integral around a closed path

(B) $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$ Maxwell-I

charge density (charge per unit volume)

$\oint \vec{E}\cdot d\vec{S} = \frac{Q}{\epsilon_0}$ ← charge enclosed — often $\vec{E}\cdot d\vec{a}$

flux thru closed surface

Gauss's theorem/law

<u>Exercise</u>: ① Show for $\vec{E}_0 = \frac{1}{4\pi\epsilon_0}\frac{q_1\,(\widehat{\vec{r}_0-\vec{r}_1})}{|\vec{r}-\vec{r}_1|^2}$,

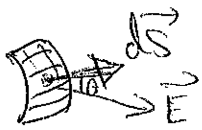
by direct computation, that $\vec{\nabla}\times\vec{E}_0 = 0$

② Look up Stoke's theorem. Using $\vec{\nabla}\times\vec{E} = 0$, use Stoke's theorem to prove ~~$\oint\vec{E}\cdot d\vec{l}$ for any~~ (A2).

~~[scribbled]~~ * Gauss's law (Gauss's dielectric flux theorem)

\# Flux through an infinitesimal ~~patch~~ patch!

$$\vec{E}\cdot d\vec{S} = E|dS|\cos\theta$$



$d\vec{S}$ points perpendicular to surface & has magnitude equal to area of patch.

$$\Phi = \int_{\Sigma} \vec{E}\cdot d\vec{S}$$ definition of flux thru surface Σ

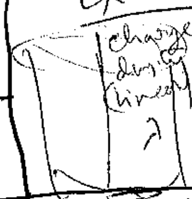
* Gauss's law is about flux thru a closed surface:

$$\boxed{\oint_{\Sigma} \vec{E}\cdot d\vec{S} = \frac{Q_{enc}}{\epsilon_0}}$$ Q_{enc} = total enclosed charge

Ex. 1

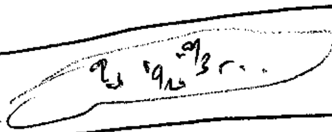
$$\oint \vec{E}\cdot d\vec{S} = \frac{q_1}{\epsilon_0}$$

Ex. 3



$$\oint \vec{E}\cdot d\vec{S} = \lambda L$$

Ex. 2



$$\oint \vec{E}\cdot d\vec{S} = \frac{\sum q_i}{\epsilon_0}$$

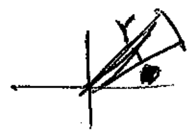
\# To prove, we will use the concept of solid angle

dA perpendicular to radial $\Big\}$ $d\Omega = \frac{dA}{r^2}$

$$\int d\Omega = 4\pi$$

(10)

Contrast to usual (planar) angle:

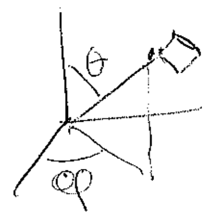


$$d\phi = \frac{dl}{r}$$

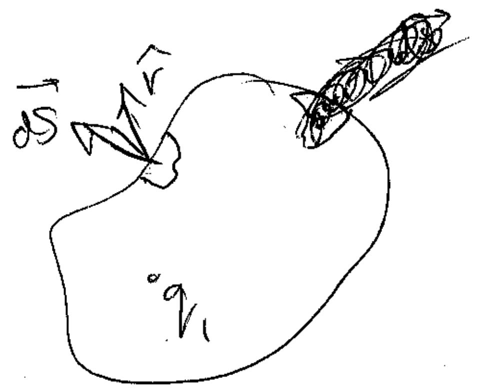
$$\int d\phi = 2\pi$$

If you know spherical coordinates,

$$d\Omega = \sin\theta \, d\theta \, d\varphi$$



Proving Gauss's law for a single point charge enclosed:



$$\oint \vec{E} \cdot d\vec{S}$$

$$= \frac{q_1}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} \cdot d\vec{S}$$

Now $\hat{r} \cdot d\vec{S} = |dS| \cos\theta = dA$ (perpendicular to radial direction)

$$\vec{E} \cdot d\vec{S} = \frac{q_1}{4\pi\epsilon_0} \, d\Omega$$

$$\oint \vec{E} \cdot d\vec{S} = \frac{q_1}{4\pi\epsilon_0} \cdot \oint d\Omega = \frac{q_1}{\epsilon_0}$$ for single POINT CHARGE.

** Using superposition principle: can extend to any collection/distribution of charges

GAUSS' LAW EXAMPLES

Ex.1 A point charge in a closed surface

$$\oint_\Sigma \vec{E}\cdot\vec{dS} = \frac{q_1}{\epsilon_0}$$

Ex.2 Multiple point charges in a closed surface

$$\oint_\Sigma \vec{E}\cdot\vec{dS} = \frac{q_1+q_2+q_3+\cdots}{\epsilon_0}$$

$$= \frac{1}{\epsilon_0}\sum_i q_i$$

Ex.3a Constant line charge. Cylinder encloses part of this line charge. Linear charge density λ

$$\oint_\Sigma \vec{E}\cdot\vec{dS} = \frac{1}{\epsilon_0}\lambda L$$

λL = charge enclosed WITHIN cylindrical closed surface

Ex. 3a Same charge, spherical surface

Length of line inside surface $= 2R$

$$\oint_{\Sigma} \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \lambda \cdot 2R$$

What if linear charge density were not constant ?

Instead of $\lambda(2R)$, you would need to integrate over line to obtain enclosed charge

Ex. 4 Charge distribution with volume charge density $\rho(\vec{r})$

$$\oint_{\Sigma} \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int_V \rho(\vec{r}) d^3r$$

$V =$ volume enclosed by Σ

Ex. 5a Point charge at centre of spherical surface.

$$\oint \vec{E} \cdot \vec{dS} = \frac{q}{\epsilon_0}$$

Did you know this already? $\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} \hat{r}$

$$\oint \vec{E} \cdot \vec{dS} = \frac{1}{4\pi\epsilon_0} \oint \frac{q}{R^2} \hat{r} \cdot \vec{dS}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} \oint dS$$

$\boxed{\hat{r} \parallel \vec{dS} \text{ everywhere!}}$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} \cdot 4\pi R^2 = \frac{q}{\epsilon_0}$$

Ex. 5b Point charge inside spherical surface, not at centre.

$$\oint \vec{E} \cdot \vec{dS} = \frac{q}{\epsilon_0}$$

Did not know this without Gauss's law.

$\vec{E} \nparallel \vec{dS}$; integral not as trivial.

* From Gauss' theorem To Maxwell's 1st eq and Poisson's Eq.

$$\oint_{\Sigma} \vec{E}\cdot d\vec{S} = \frac{Q}{\epsilon_0} = \frac{1}{\epsilon_0}\int_V \rho\, dV$$

Here V is the volume enclosed by Σ

(Assuming continuous charge dist^n

Notation: $dV \equiv d^3r \equiv dx\,dy\,dz$
$\equiv r^2 \sin\theta\, dr\, d\theta\, d\varphi$

We use Gauss's divergence theorem!

$$\oint_{\Sigma} \vec{A}\cdot d\vec{S} = \int_V (\vec{\nabla}\cdot\vec{A})\, dV$$

Exercise! Review the divergence thm and Stoke's theorem.

$$\Rightarrow \int_V (\vec{\nabla}\cdot\vec{E})\, dV = \int_V \rho\, dV$$

This is true for arbitrary volume V.

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}} \longrightarrow \text{Maxwell I}$$

Recalling $\vec{E} = -\vec{\nabla} V$,

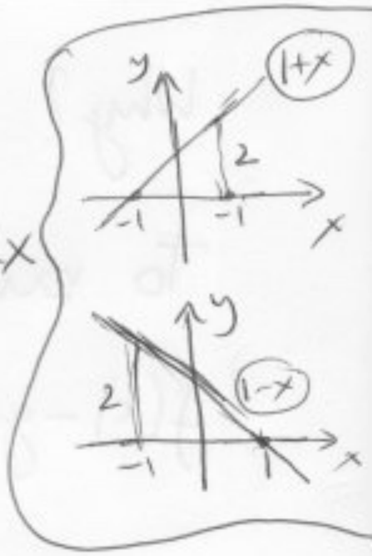
$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}} \quad \text{Poisson's equation.}$$

* Integrals equal $\underset{??}{\Longrightarrow}$ integrands equal ?

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$$

If true for a particular (a,b), does NOT imply $f(x) = g(x)$

Example $\displaystyle\int_{-1}^{1} (1-x)\,dx = \int_{-1}^{1} (1+x)\,dx$

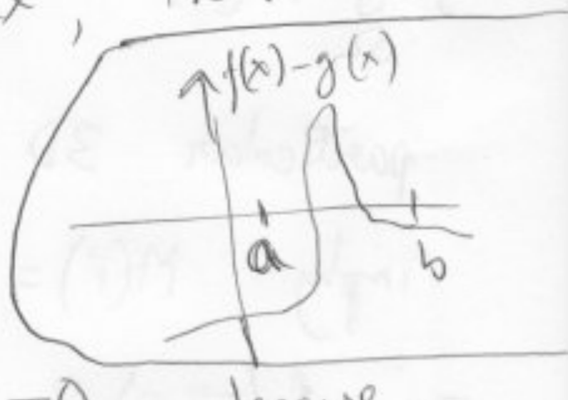


$$\not\Rightarrow \quad 1-x = 1+x$$

Integrals cannot be canceled.

* If $\displaystyle\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$, then

$$\int_a^b \left[f(x) - g(x)\right] dx = 0$$



Does not imply $f(x) - g(x) = 0$, because integral can vanish even if $f(x)-g(x)$ is nontrivial, nonzero function

11b

If $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$ for

ANY interval (a,b), then $f(x) = g(x)$.

Why? If $\int_a^b \left[f(x) - g(x)\right] dx$ has

to vanish for ANY (a,b), then

$f(x) - g(x)$ has to vanish everywhere.

* Similar for 3D integrals.

If $\int_{\Omega_0} M(\vec{r})\, d^3r = \int_{\Omega_0} N(\vec{r})\, d^3r$ for a

particular 3D region Ω_0, does NOT

imply $M(\vec{r}) = N(\vec{r})$

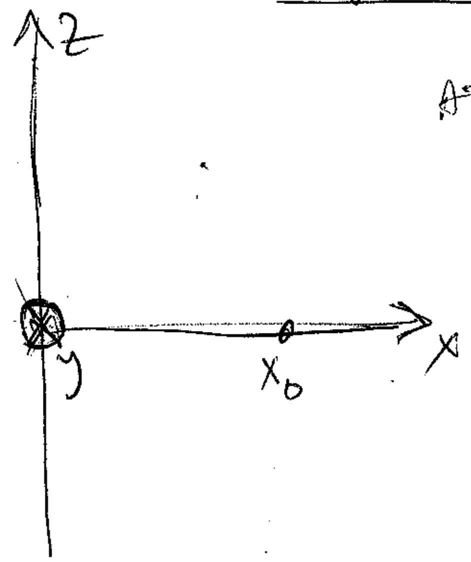
If $\int_V \left(\vec{\nabla}\cdot\vec{E}\right) d^3r = \int_V \left(\frac{\rho}{\epsilon_0}\right) d^3r$ for ANY

3D region V, we must have $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$

* Applications of Gauss's law.

Applicatⁿ-IIA

Infinite thin rod → Done in assignments using integration.



At $(x_0,0,0)$

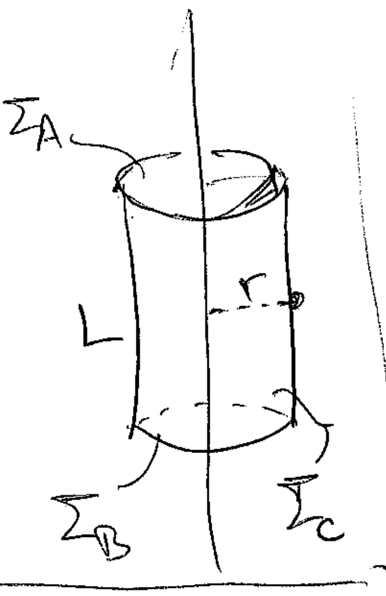
$$\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{x_0}\hat{i}$$

At a general point, using Cylindrical coordinates

$$\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}\hat{r}$$

(Here r is the cylindrical coordinate.)

Using Gauss's law



$$\boxed{\Sigma = \Sigma_A \cup \Sigma_B \cup \Sigma_C}$$

symmetry:

Same $|\vec{E}|$, pointing outward, at all points of the curved part of cylinder.

$$\int_{\Sigma_C} \vec{E}\cdot d\vec{S} = |\vec{E}|\int_{\Sigma_C} |d\vec{S}|$$

$$= |\vec{E}| \times 2\pi r L$$

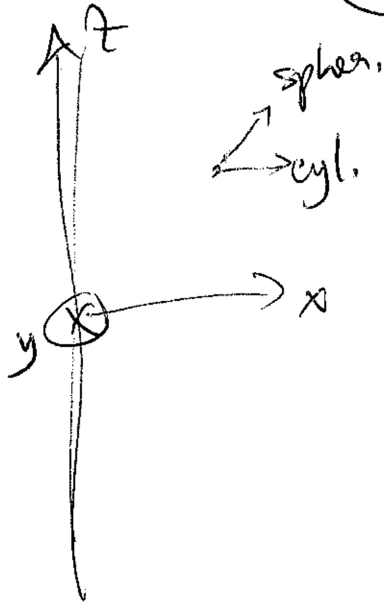
$$\int_{\Sigma_A} \vec{E}\cdot d\vec{S} = \int_{\Sigma_B} \vec{E}\cdot d\vec{S} = 0$$

$$\int_{\Sigma} \vec{E}\cdot d\vec{S} = 0 + 0 + |\vec{E}|\,2\pi r L \overset{\text{Gauss's thm}}{=} \frac{\lambda L}{\epsilon_0}$$

$$E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}$$

For thin charged rod,

$$\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}\hat{r}$$

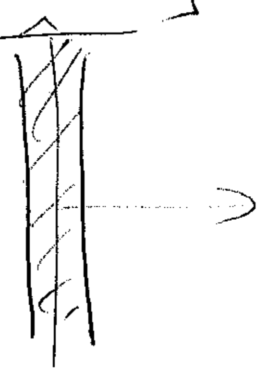


<u>CAUTION</u> $\vec{r}, r, \hat{r}$

is a cylindrical coordinate/vector, not a spherical one.

$\boxed{\begin{array}{l}\text{* \underline{A CONDUCTOR}: conducts charge.} \\ \text{• Static situation} \rightarrow \text{all charge sits at surface.}\end{array}}$

Infinite <u>thick</u> rod of charge:



when metallic (conducting)

Top view of charge density

When uniformly charged

Top view of charge density

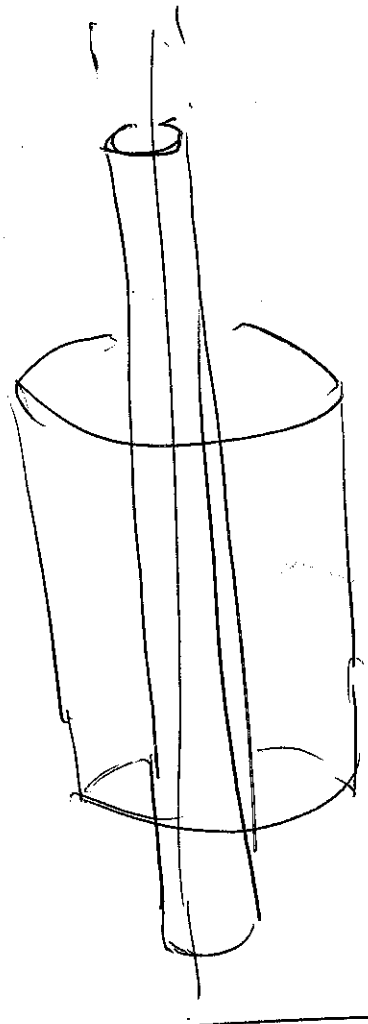
Could also consider HOLLOW rod

→ For electric field OUTSIDE, internal structure doesn't matter

Applicatⁿ - IB Infinitely long cylinder — either metallic or filled with charge uniformly.

charge resides on outside

Charge λ per unit length.



For electric field outside the rod, exactly the same argument holds.

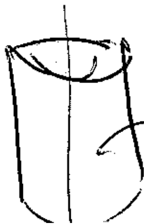
$$\int_{\Sigma} \vec{E} \cdot dS = \frac{\lambda L}{\epsilon_0}$$

$$\Rightarrow E = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{r}$$

As if it was a thin rod

Application 1 C

Inside an infinite hollow charged cylinder (or metallic cylinder)

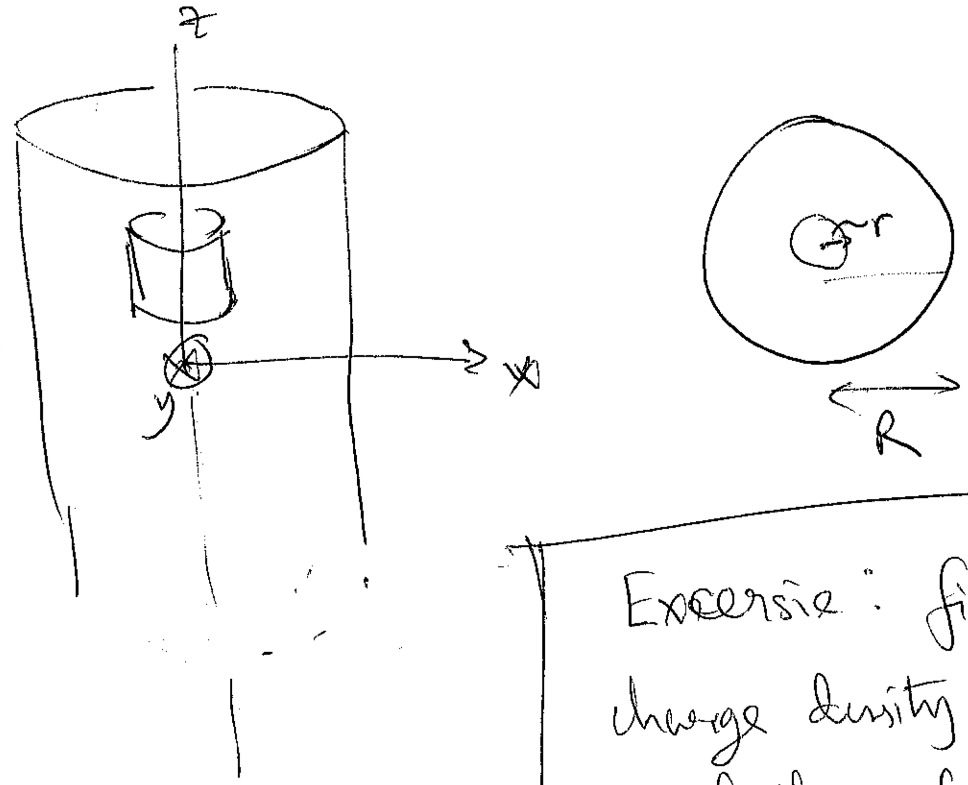


E=0

Exercise! Construct smaller cylinder & show that E=0

Applicatⁿ 1D

Thick rod uniformly filled with charge.
CYLINDRICALLY symmetric. Charge density ρ.
Calculate $\vec{E}$ inside the rod:



Excersie: find the linear charge density of the rod, as function of R and ρ

$\vec{E}$ should depend on r only, and point radially away from axis.

$$\oint_{\Sigma} \vec{E}\cdot d\vec{S} = \oint_{\text{curved}} \vec{E}\cdot d\vec{S} + \oint_{\text{caps}} \vec{E}\cdot d\vec{S}$$

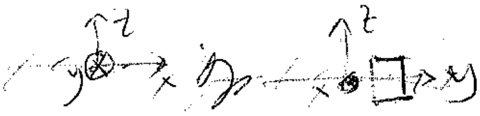
$$= E \cdot 2\pi r L + 0$$

$$E \cdot 2\pi r L = \frac{Q}{\epsilon_0} = \frac{1}{\epsilon_0} \rho \pi r^2 L$$

$$\Rightarrow E = \frac{\rho}{2\epsilon_0} \cancel{2} r$$

Exercise: Plot E as a function of r, clearly differentiating $r < R$, $r = R$ and $r > R$.

Application 2



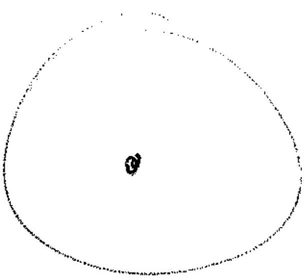
Infinite sheet of charge (Problem Set 03)

$$E \cdot A + E \cdot A = \frac{Q}{\epsilon_0}$$

$$\Rightarrow 2EA = \frac{\sigma A}{\epsilon_0} \Rightarrow \boxed{E = \frac{\sigma}{2\epsilon_0}}$$

Application 3a

Point charge Q



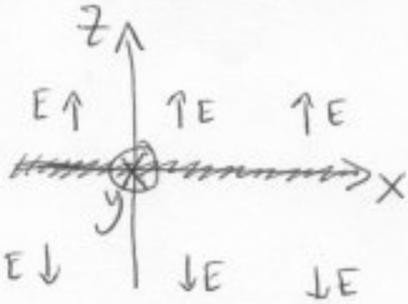
$$\oint \vec{E} \cdot d\vec{S} = E \cdot 4\pi r^2 = \frac{Q}{\epsilon_0}$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$$

→ "circular" derivation: we derived Gauss's th^m from the eq., so of course Gauss's th^m gives it back.

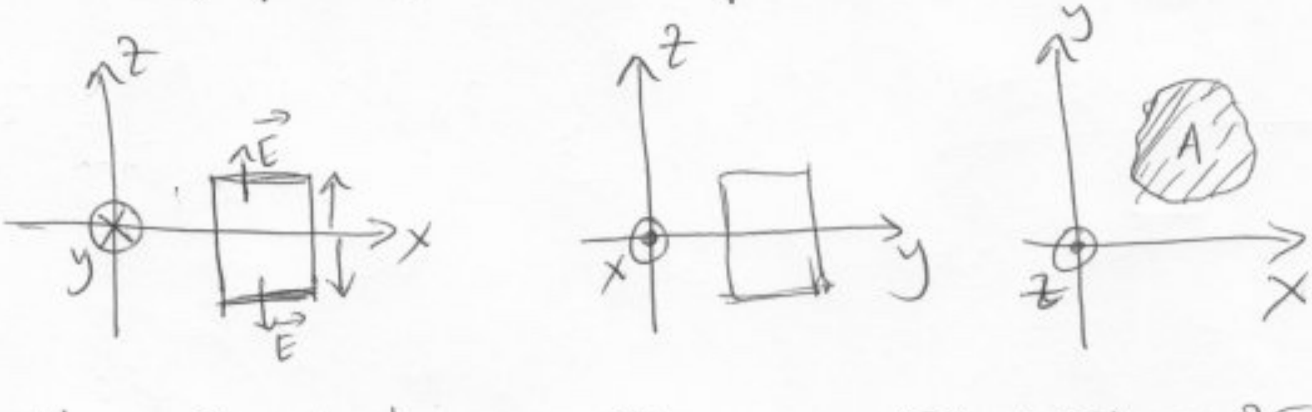
Application 2 — Infinite sheet of charge

XY-plane carries uniform surface charge; surf. charge density σ.



By symmetry, if σ is positive, $\vec{E}$-field points in $+\hat{k}$ direction for $z>0$ and in $-\hat{k}$ direction for $z<0$.

Choose Gaussian surface with two sides parallel to xy plane; the rest parallel to z-axis



Use Gauss' law. Flux $= EA + EA = 2EA$

Charge enclosed $= \sigma A$

$$2EA = \frac{\sigma A}{\epsilon_0} \Rightarrow \boxed{E = \frac{\sigma}{2\epsilon_0}}$$ Indep. of distance!!