Electricity and magnetism: an introduction to Maxwell's equations

MP204

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Preface

These lecture notes accompany a course which is a short introduction to the four famous *Maxwell equations*. These four equations unify electric and magnetic phenomena and give birth to what is thereafter called the *electromagnetic field.*

Maxwell gave a lecture on his work to the Royal Society of London in 1864 and his results were then published[1] in 1865. Faraday had earlier suggested[2] that light was as an electromagnetic wave in 1846; this fact was duly acknowledged by Maxwell in his paper.

There are a huge number of books on electromagnetic theory and so we only recommend three; the college library will provide one with many, many more. So our three titles are—the first one is the main text, the others are for subsidiary reading:

1. Grant I. S. and Philips W. R., *Electromagnetism*, Wiley, (1990).

2. Feynman R. P., Leighton R. B. and Sands M. L., *The Feynman lectures on physics: volume II*, Addison–Wesley, (1965).

3. Purcell E.M., *Electricity and Magnetism, Berkeley physics course volume II*, McGraw–Hill, (1985).

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[1] Maxwell J. C., *A dynamical theory of the electromagnetic field*, Phil. Tran. Roy. Soc, **155**, 459–512, (1865).

[2] Faraday M., *Thoughts on ray vibrations*, Phil. Mag., **28**, 345–350, (1846). Faraday's *Thoughts on ray vibrations*, were actually delivered in 1846 as an off the cuff lecture to the Royal Society on the occasion of the scheduled speaker not being available.

It seems very likely that Faraday was stimulated to think along these lines by the fact that in 1845 he had carried out an experiment which showed that polarised light had its plane of polarisation rotated when it passed through a magnetic field.

CHAPTER I

From electric charges to potentials

§ 1. Preliminaries on constants and units

IN this course we shall provide material which is intended to be self-contained but reference elsewhere is occasionally needed. It is also good practice to read around a subject as widely as one's time will allow and, in particular, to look at the books recommended in the preface.

The units we shall use are MKSA units which are the standard units currently used in all the natural sciences and in engineering. For electromagnetism they are not ideal since (as we shall gradually see) their definitions contain arbitrary factors of π in situations where there is no circular, cylindrical or spherical symmetry. The most unfortunate consequence of this (as will become obvious at the time) factors of π then disappear from situations where there is some circular, cylindrical or spherical symmetry.

Finally a universal constant that occurs in electromagnetism is ϵ_0 known as the *permittivity of free space*[1] . In our MKSA units its value is given by

$$\begin{aligned} \epsilon_0 &= 8.85 \times 10^{-12} \text{ coulomb}^2/\text{newton-metre}^2 \\ \text{or entirely equivalently } \epsilon_0 &= 8.85 \times 10^{-12}\text{volt-metre/coulomb} \end{aligned} \tag{1.1}$$

The coulomb being the unit of charge as we shall see below. It is often useful, for numerical purposes, to know that, since c, the velocity of light is given by

$$c = 3 \times 10^8 \text{m/sec} \tag{1.2}$$

then

$$\begin{aligned} \epsilon_0 c^2 &= \frac{10^7}{4\pi} \\ \Rightarrow \frac{1}{4\pi\epsilon_0} &\simeq 9 \times 10^9 \end{aligned} \tag{1.3}$$

[1] The phrase *free space* often occurs in electromagnetic theory and it refers to *charge free space* i.e. a vacuum as opposed to a solid, liquid or gas.

§ 2. Coulomb's law.

The fundamental fact lying at the base of all electromagnetism is that the forces between charges are of the inverse square type. The formal statement of this fact is known as *Coulomb's law.* Formally we have

Coulomb's law *If two charges of size* q_1 *and* q_2 *are located at* $\mathbf{r}_1$ *and* $\mathbf{r}_2$ *respectively then the force* $\mathbf{F}$ *between them is given by*

$$\begin{aligned}\mathbf{F} &= \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\mathbf{r}_1-\mathbf{r}_2|^2}(\widehat{\mathbf{r}_1-\mathbf{r}_2})\\ &= \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}(\mathbf{r}_1-\mathbf{r}_2)\end{aligned} \tag{1.4}$$

When the force is computed with this formula the units of charge are called *Coulomb's.*

This law implies the familiar property that like charges repel and that unlike charges attract. With Coulomb's law under our belt we can immediately proceed to the notion of an electric field $\mathbf{E}$.

Definition (Electric field $\mathbf{E}$) *The electric field* $\mathbf{E}$ *at a point* $\mathbf{r}$ *exerted by any collection of charges is the force that would act on a unit charge placed at* $\mathbf{r}$*.*

Example *The electric field due to a single charge*

Combining this definition with Coulomb's law we can immediately compute the electric field $\mathbf{E}(\mathbf{r})$ at $\mathbf{r}$ exerted by a *single charge* q. For, if q is located at $\mathbf{r}_1$, then the force $\mathbf{F}$ on a unit charge at $\mathbf{r}$ is given by

$$\begin{aligned}\mathbf{F} &= \frac{q}{4\pi\epsilon_0}\frac{(\widehat{\mathbf{r}-\mathbf{r}_1})}{|\mathbf{r}-\mathbf{r}_1|^2}\\ \text{i.e. } \mathbf{E}(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0}\frac{(\widehat{\mathbf{r}-\mathbf{r}_1})}{|\mathbf{r}-\mathbf{r}_1|^2}\end{aligned} \tag{1.5}$$

It now easily follows that if a charge Q is placed in an electric field $\mathbf{E}$ then it is acted on (at $\mathbf{r}$) with a force $\mathbf{F}(\mathbf{r})$ where

$$\mathbf{F}(\mathbf{r}) = Q\,\mathbf{E}(\mathbf{r}) \tag{1.6}$$

and if the precise point $\mathbf{r}$ meant is not important, we often abbreviate this to simply

$$\mathbf{F} = Q\,\mathbf{E} \tag{1.7}$$

Having obtained the electric field exerted by one charge we now want the field due to a collection of charges. For this purpose we need what is called the *principle of superposition.* This is an *experimentally discovered fact* which says roughly that electric fields due to separate charges "add together". More formally we have

The principle of superposition. *If* n *charges* $q_1, q_2, \ldots, q_n$ *are located at* $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ *respectively, then their electric field* $\mathbf{E}$ *at* r *is additive, i.e. it is given by*

$$\mathbf{E}(\mathbf{r}) = \frac{q_1}{4\pi\epsilon_0}\frac{(\widehat{\mathbf{r}-\mathbf{r}_1})}{|\mathbf{r}-\mathbf{r}_1|^2} + \frac{q_2}{4\pi\epsilon_0}\frac{(\widehat{\mathbf{r}-\mathbf{r}_2})}{|\mathbf{r}-\mathbf{r}_2|^2} + \cdots + \frac{q_n}{4\pi\epsilon_0}\frac{(\widehat{\mathbf{r}-\mathbf{r}_n})}{|\mathbf{r}-\mathbf{r}_n|^2} \tag{1.8}$$

More briefly we can write

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 + \cdots + \mathbf{E}_n \\ &= \sum_{i=1}^{n} \mathbf{E}_i \\ \text{where}\ \ \mathbf{E}_i &= \frac{q_i}{4\pi\epsilon_0} \frac{(\widehat{\mathbf{r}-\mathbf{r}_i})}{|\mathbf{r}-\mathbf{r}_i|^2}\end{aligned} \tag{1.9}$$

The electric field, being a vector quantity, requires three quantities for its specification; actually this information triplet contains a lot of redundancy. We shall now see that only one scalar quantity is really needed to specify an electric field. this quantity is known as *the potential* and it is the next thing that we shall consider.

§ 3. The potential function V or Φ

There is a potential function V (also often denoted by Φ) associated with every electric field $\mathbf{E}$. For a given $\mathbf{E}$ it is defined by the equation

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= -\operatorname{grad} V(\mathbf{r}) \\ &\equiv -\nabla V(\mathbf{r}) \\ &= -\left(\frac{\partial V(\mathbf{r})}{\partial x}\mathbf{i} + \frac{\partial V(\mathbf{r})}{\partial y}\mathbf{j} + \frac{\partial V(\mathbf{r})}{\partial z}\mathbf{k}\right)\end{aligned} \tag{1.10}$$

where of course

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{1.11}$$

Example *The potential for a single charge*

If we place a single charge of size q_1, say, at the location $\mathbf{r}_1$then it is easy to check by direct differentiation that its potential at an arbitrary point $\mathbf{r}$ is given by

$$V(\mathbf{r}) = \frac{q_1}{4\pi\epsilon_0} \frac{1}{|\mathbf{r}-\mathbf{r}_1|} \tag{1.12}$$

i.e. one has that the electric field $\mathbf{E}$ of the charge, which we obtained above in 1.5 , is given by

$$\mathbf{E} = -\nabla \left(\frac{q_1}{4\pi\epsilon_0} \frac{1}{|\mathbf{r}-\mathbf{r}_1|}\right) \tag{1.13}$$

and indeed the differentiation gives us the result that

$$-\nabla \left(\frac{q_1}{4\pi\epsilon_0} \frac{1}{|\mathbf{r}-\mathbf{r}_1|}\right) = -\frac{q_1}{4\pi\epsilon_0} \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r}_1|}\right) = \frac{q_1}{4\pi\epsilon_0} \frac{(\widehat{\mathbf{r}-\mathbf{r}_1})}{|\mathbf{r}-\mathbf{r}_1|^2} \tag{1.14}$$

in perfect agreement with 1.5.

The principle of superposition for potentials *It is easy to prove that the principle of superposition also applies to potentials. To prove this assume that we have, as in 1.8, n charges $q_1, q_2, \ldots, q_n$ at $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ respectively, then their potential $V(\mathbf{r})$ at $\mathbf{r}$ is given by*

$$\begin{aligned} V(\mathbf{r}) &= V_1(\mathbf{r}) + V_2(\mathbf{r}) + \cdots + V_n(\mathbf{r}) \\ &= \sum_{i=1}^{n} V_i(\mathbf{r}) \\ \textit{where} \quad V_i(\mathbf{r}) &= \frac{q_i}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_i|} \end{aligned} \tag{1.15}$$

This is easy to prove: we know know that the electric field produced by these charges is given by

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 + \cdots + \mathbf{E}_n \\ &= \sum_{i=1}^{n} \mathbf{E}_i \\ \text{where} \quad \mathbf{E}_i &= \frac{q_i}{4\pi\epsilon_0} \frac{(\widehat{\mathbf{r}-\mathbf{r}_i})}{|\mathbf{r} - \mathbf{r}_i|^2} \end{aligned} \tag{1.16}$$

but we also know that

$$\mathbf{E}_i = -\nabla \left(\frac{q_i}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_i|} \right) \tag{1.17}$$

Hence we can write

$$\begin{aligned} \mathbf{E} &= -\nabla \left(\frac{q_1}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_1|} \right) - \nabla \left(\frac{q_2}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_2|} \right) - \cdots - \nabla \left(\frac{q_n}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_n|} \right) \\ &= -\nabla \left(\frac{q_1}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_1|} + \frac{q_2}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_2|} + \cdots + \frac{q_n}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_n|} \right) \end{aligned} \tag{1.18}$$

In other words we have

$$\mathbf{E} = -\nabla V \tag{1.19}$$

where

$$\begin{aligned} V(\mathbf{r}) &= V_1(\mathbf{r}) + V_2(\mathbf{r}) + \cdots + V_n(\mathbf{r}) \\ \text{and} \quad V_i(\mathbf{r}) &= \frac{q_i}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_i|} \end{aligned} \tag{1.20}$$

which is indeed the principle of superposition.

Now we see that given a particular potential V it is easy to find the corresponding electric field $\mathbf{E}$: one just has to do the differentiations appropriate for the expression $-\nabla V$. We would like to be able to go in the reverse direction: i.e. given the electric field $\mathbf{E}$ construct its associated potential V. This is indeed possible but is a little harder as, can easily be anticipated, it involves *integration* rather than *differentiation*. We are ready to digest the argument.

The argument rests on one technical piece of calculus. This is that if $f(x, y, z)$ is any differentiable function, then

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \tag{1.21}$$

which follows from Taylor's theorem. It also follows that

$$\int df = f \tag{1.22}$$

Suppose now that we are given an electric field $\mathbf{E}$. Let V be its potential function so that

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \tag{1.23}$$

But if we write

$$\mathbf{dr} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \tag{1.24}$$

then we note that

$$\begin{aligned} \nabla V \cdot \mathbf{dr} &= \left(\frac{\partial V(\mathbf{r})}{\partial x} + \frac{\partial V(\mathbf{r})}{\partial y} + \frac{\partial V(\mathbf{r})}{\partial z} \right) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\ &= \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \\ &= dV \end{aligned} \tag{1.25}$$

In other words we have

$$dV = -\mathbf{E} \cdot \mathbf{dr} \tag{1.26}$$

Finally we take a path Γ, say, beginning at an arbitrary but *fixed* point $\mathbf{r}_0$ and ending at $\mathbf{r}$ and we integrate along Γ. In this way we obtain

$$\begin{aligned} \int_\Gamma dV \equiv \int_{\mathbf{r}_0}^{\mathbf{r}} dV &= -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot \mathbf{dr} \\ \Rightarrow V(\mathbf{r}) - V(\mathbf{r}_0) &= -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot \mathbf{dr} \\ \Rightarrow V(\mathbf{r}) &= V(\mathbf{r}_0) - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot \mathbf{dr} \end{aligned} \tag{1.27}$$

But we can discard the constant quantity $V(\mathbf{r}_0)$ on the right hand side of the last equation since we can always alter a potential V by a constant without changing its associated electric field: this is obvious if one simply notes that, if C is a constant, then

$$\nabla(V + C) = \nabla V, \quad \text{because } \nabla C = 0 \tag{1.28}$$

Thus we take our final expression for the potential V due to an electric field $\mathbf{E}$ to be simply

$$V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot \mathbf{dr} \tag{1.29}$$

Summarising the relations between $\mathbf{E}$ and V then gives us the pair of equations

$$\begin{aligned} \mathbf{E} &= -\nabla V \\ V &= -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot \mathbf{dr} \end{aligned} \tag{1.30}$$

§ 4. Laplace's equation

Our next topic will be to obtain an important equation due to Laplace and others which is obeyed by V. The potential V due to any (discrete) system of charges satisfies an equation known as *Laplace's equation.* This equation is

$$\nabla^2 V = 0 \tag{1.31}$$

or, spelled out in more detail,

$$\text{div} \cdot \text{grad}\, V = \nabla \cdot (\nabla V) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V = 0 \tag{1.32}$$

The proof of Laplace's equation is not difficult; because of the superposition principle it requires just one simple calculation involving the potential due to a single charge. We now give the proof: take a general collection of n charges so that, as in 1.20, their potential at $\mathbf{r}$ is given by

$$\begin{aligned} V(\mathbf{r}) &= V_1(\mathbf{r}) + V_2(\mathbf{r}) + \cdots + V_n(\mathbf{r}) \\ \text{and} \quad V_i(\mathbf{r}) &= \frac{q_i}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_i|} \end{aligned} \tag{1.33}$$

Hence

$$\begin{aligned} \nabla^2 V &= \nabla^2 \left(V_1(\mathbf{r}) + V_2(\mathbf{r}) + \cdots + V_n(\mathbf{r}) \right) \\ &= \sum_{i=1}^{n} \nabla^2 V_i(\mathbf{r}) \\ &= 0, \quad \text{since, as we shall now show,} \\ \nabla^2 V_i(\mathbf{r}) &= 0, \quad \text{for each } i \end{aligned} \tag{1.34}$$

All that remains is to show that

$$\nabla^2 V_i(\mathbf{r}) = 0 \tag{1.35}$$

and we do this by direct differentiation. We simply write

$$\begin{aligned} \mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ \mathbf{r}_i &= x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k} \end{aligned} \tag{1.36}$$

So that

$$\begin{aligned} |\mathbf{r} - \mathbf{r}_i| &= |\{(x - x_i)\mathbf{i} + (y - y_i)\mathbf{j} + (z - z_i)\mathbf{k}\}| \\ &= \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2} \end{aligned} \tag{1.37}$$

This means that

$$\begin{aligned} \nabla^2 V_i(\mathbf{r}) &= \frac{q_i}{4\pi\epsilon_0}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) A \\ \text{where } A &= \left(\frac{1}{\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}}\right) \end{aligned} \tag{1.38}$$

and it is then a completely straightforward matter to verify that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}}\right) = 0 \tag{1.39}$$

So we have indeed proved Laplace's equation for an arbitrary *discrete* collection of charges.

It is worthwhile observing that Laplace's equation is really a consequence of the Coulomb's inverse square law of force. Hence the potential V for other situations where an inverse square law applies will also satisfy Laplace's equation. For example the *gravitational potential* produced by a system of masses also satisfies Laplace's equation.

This brings the present chapter to a close.

CHAPTER II

Calculating electric fields: Gauss's theorem

§ 1. Gauss' dielectric flux theorem

WE are now ready to consider a remarkable result whose existence is directly traceable to the inverse square force law between charges—were this force law to be an inverse *cube* law, or indeed were this force to decrease with distance at any rate *other* than an inverse square, then Gauss' dielectric flux theorem would not hold but would be replaced by something more complicated. The theorem states the following

Theorem (Gauss' dielectric flux theorem) *If a closed surface S contains a total amount of electric charge Q then the flux $\int_S \mathbf{E}\cdot\mathbf{dS}$ of the electric field $\mathbf{E}$ out of S is given by*

$$\int_S \mathbf{E}\cdot\mathbf{dS} = \frac{Q}{\epsilon_0} \tag{2.1}$$

Proof: We shall give a proof which is valid for a *discrete* collection of charges. Hence we shall now assume that Q is a collection of a finite number n of charges q_1, q_2, …, q_n.

Now let us take just one of these charges q_i, say, which is located at the point O. Now consider an arbitrary infinitesimal patch $\mathbf{dS}$ on the surface S which is a distance r from O. The flux of q_i through this patch is

$$\mathbf{E}\cdot\mathbf{dS} \tag{2.2}$$

But $\mathbf{E}$ on $\mathbf{dS}$ is given by

$$\mathbf{E} = \frac{q_i}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2} \tag{2.3}$$

so that

$$\mathbf{E}\cdot\mathbf{dS} = \frac{q_i}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\mathbf{dS}}{r^2} \tag{2.4}$$

However this flux is simply related to a certain solid angle as follows: the solid angle subtended by $\mathbf{dS}$ at O is $d\Omega$ where

$$d\Omega = \frac{dA}{r^2} \tag{2.5}$$

and dA denotes the area of a spherical cap normal to $\mathbf{E}$, cf. diagram. This means that we have

$$dA = |\mathbf{dS}| \cos\theta \tag{2.6}$$

where θ is the angle between $\mathbf{E}$ and $\mathbf{dS}$. Since we also have

$$\hat{\mathbf{r}} \cdot \mathbf{dS} = |\mathbf{dS}| \cos\theta \tag{2.7}$$

then we have the equation

$$\mathbf{E} \cdot \mathbf{dS} = \frac{q_i}{4\pi\epsilon_0} d\Omega \tag{2.8}$$

We now immediately integrate to obtain

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \frac{q_i}{4\pi\epsilon_0} \int_S d\Omega \tag{2.9}$$

But it is an elementary fact that

$$\int_S d\Omega = 4\pi \tag{2.10}$$

Hence we have proved that

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \frac{q_i}{\epsilon_0} \tag{2.11}$$

Now all we have to do is to sum both sides of 2.11 over i so as to include all charges; all this does is to replace the q_i on the RHS by the total charge Q giving us the desired result

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \frac{Q}{\epsilon_0} \tag{2.12}$$

and the proof is complete.

§ 2. Maxwell's first equation and Poisson's equation

We are now ready to derive Maxwell's first equation which is simply

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{2.13}$$

One starts with Gauss's flux theorem

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \frac{Q}{\epsilon_0} \tag{2.14}$$

then we suppose that the charge Q comes totally from charge inside, or on the surface of, a closed volume V whose boundary is the surface S above. This means that, if ρ is the charge density per unit volume, then

$$Q = \int_V \rho \, dV \tag{2.15}$$

so that we have immediately that

$$\int_S \mathbf{E}\cdot\mathbf{dS} = \frac{1}{\epsilon_0}\int_V \rho\, dV \tag{2.16}$$

Gauss's divergence theorem applied to the LHS then yields the equation

$$\int_V \nabla\cdot\mathbf{E}\, dV = \int_V \frac{\rho}{\epsilon_0}\, dV \tag{2.17}$$

But since the volume V is arbitrary then the integrands on both sides of 2.17 must be equal; hence we have

Maxwell's first equation

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \text{Maxwell's first equation} \tag{2.18}$$

as desired.

If we write this equation in terms of the potential V rather than the electric field $\mathbf{E}$ then the equations that we get is called *Poisson's equation*: recalling that

$$\mathbf{E} = -\nabla V \tag{2.19}$$

we substitute for $\mathbf{E}$ in 2.17 and obtain an equation for V which is

$$\begin{aligned} \nabla\cdot(-\nabla V) &= \frac{\rho}{\epsilon_0} \\ \Rightarrow \nabla^2 V &= -\frac{\rho}{\epsilon_0} \end{aligned} \tag{2.20}$$

Poisson's equation is the last of the two equations above, that is the following equation for V

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}, \quad \text{(Poisson's equation)} \tag{2.21}$$

§ 3. Gauss's theorem at work

Gauss's theorem is a very useful tool for calculating the electric field in a variety of situations. We shall now consider some examples of this.

Example *The electric field due to a sphere of charge*

Our task now is to compute the electric field created by a sphere of charge, where the total charge on the sphere is Q. We suppose that there exists a *solid* sphere of charge, of radius a and centre O; we then wish to calculate the electric field $\mathbf{E}$ at an arbitrary point P where P is a distance r from O. We also assume that $r > a$ so that P is a point *outside* the sphere. Later we shall show how to compute $\mathbf{E}$ when P is *inside* the sphere.

The technique used is just a judicious use of Gauss's theorem

$$\int_S \mathbf{E}\cdot\mathbf{dS} = \frac{Q}{\epsilon_0} \tag{2.22}$$

The key matter is to choose the right surface S over which to integrate $\mathbf{E}$. We take S to be a sphere of radius r and *centre* O so that it is concentric to the sphere of charge.

Now we can deduce that spherical symmetry demands that $\mathbf{E}$ be *radial* on the surface of S, i.e. $\mathbf{E}$ is parallel to $\mathbf{dS}$ on S since $\mathbf{dS}$, by definition, is always radial. Hence we have

$$\int_S \mathbf{E}\cdot\mathbf{dS} = \int_S |\mathbf{E}||\mathbf{dS}| \tag{2.23}$$

But all points on S are equidistant from O so $|\mathbf{E}|$ must be constant on S, therefore we can write

$$\begin{aligned}\int_S |\mathbf{E}||\mathbf{dS}| &= |\mathbf{E}|\int_S |\mathbf{dS}| \\ &= |\mathbf{E}|4\pi r^2\end{aligned} \tag{2.24}$$

where we have used the obvious fact that $\int_S |\mathbf{dS}|$ is just the total surface area of S. We have now deduced that

$$|\mathbf{E}|4\pi r^2 = \frac{Q}{\epsilon_0} \tag{2.25}$$

and since we already know that $\mathbf{E}$ is radial we have the complete expression for $\mathbf{E}$ which is

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{|\mathbf{r}|^2} \tag{2.26}$$

It is noteworthy that this expression expresses the eminently reasonable fact that a sphere of charge behaves as if all the charge is concentrated at its centre.

Example *The electric field inside a hollow charged closed conductor*

Now let us take a *hollow* conductor and place an arbitrary charge distribution on its surface. Provided we consider points *outside* the conductor then this makes no difference to calculations of the electric field which use Gauss's theorem. However, since the conductor is hollow we can now go inside and, for points *inside*, the situation is radically different. In fact the electric field is always *zero* for all points inside a hollow closed charged conductor.

We shall not give a general proof [1] of the above facts but shall prove them for the case when the conductor is spherical of radius a.

First we choose a point P outside the conductor. In this case there is nothing new to say the field $\mathbf{E}$ is exactly the same as before and given by the expression

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{|\mathbf{r}|^2} \tag{2.27}$$

where Q is the total charge on the conductor.

Next we suppose that the point P at which we want the electric field is inside the sphere. To this end let the distance from the centre of the sphere to P be r where $r < a$.

[1] The general proof follows fairly easily from the fact that a solution of Poisson's equation for the electric field of an arbitrary charge distribution is uniquely specified by the values of the potential on some closed surface (in this case the closed conducting surface).

Then we choose S to be the sphere of radius r centre O and apply Gauss's theorem from which we obtain the result

$$|\mathbf{E}|4\pi r^2 = \frac{Q}{\epsilon_0} \tag{2.28}$$

where $\mathbf{E}$ is the field at P and Q is the charge inside S. But Q has to be zero since we are inside a hollow conductor hence we immediately deduce that

$$\begin{aligned} |\mathbf{E}| &= 0 \\ \Rightarrow \mathbf{E} &= 0 \end{aligned} \tag{2.29}$$

as claimed.

Example *The electric field due to an infinite cylinder of charge*

In this example we shall compute the electric field $\mathbf{E}$ a distance r from the axis of an infinitely long charged cylinder of radius a; we shall assume that the cylinder carries a charge of λ per unit length.

This is another application of Gauss's theorem and all we have to do is to make a sensible choice for the surface S that appears in the statement of the theorem.

Cylindrical symmetry makes it reasonable that we should choose S to be a cylinder coaxial to the first, but of radius r and length L, and placed so that the point P lies on its surface. We remind the reader that Gauss's theorem requires S to be *closed* so that this cylinder consists of a curved piece of area $2\pi rL$ plus two circular discs each of area πr^2.

Applying the theorem we have

$$\int_S \mathbf{E}\cdot\mathbf{dS} = \frac{Q}{\epsilon_0} \tag{2.30}$$

but Q must be the charge inside a length L of the charged cylinder so that

$$Q = \lambda L \tag{2.31}$$

Also, if we break the integral up into two natural pieces, we get

$$\int_S \mathbf{E}\cdot\mathbf{dS} = \int_{\text{curved part}} \mathbf{E}\cdot\mathbf{dS} + \int_{\text{two discs}} \mathbf{E}\cdot\mathbf{dS} \tag{2.32}$$

Now cylindrical symmetry means that $\mathbf{E}$ must point in the *radial* direction; hence on the two discs $\mathbf{E}$ is *perpendicular* to $\mathbf{dS}$, while on the curved part $\mathbf{E}$ is *parallel* to $\mathbf{dS}$. These two observations mean that

$$\begin{aligned} \int_{\text{two discs}} \mathbf{E}\cdot\mathbf{dS} &= 0 \\ \int_{\text{curved part}} \mathbf{E}\cdot\mathbf{dS} &= \int_{\text{curved part}} |\mathbf{E}||\mathbf{dS}| \\ &= |\mathbf{E}| \int_{\text{curved part}} |\mathbf{dS}| \\ &= |\mathbf{E}|2\pi rL \end{aligned} \tag{2.33}$$

where, in the second integral, we have used the fact that all points on the curved side are equidistant from the axis of the cylinder so that $|\mathbf{E}|$ must be constant throughout the integral. We now have computed both the LHS and RHS of the expressions entering Gauss's theorem and, using these computations, we find that

$$|\mathbf{E}|2\pi rL = \frac{\lambda L}{\epsilon_0} \tag{2.34}$$

We deduce at once that

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{|\mathbf{r}|} \tag{2.35}$$

and it is useful to remember that the cylindrical geometry has rendered $|\mathbf{E}|$ proportional to $1/r$ rather than $1/r^2$.

Example *The electric field due to an infinite plane of charge*

Perhaps the easiest example, though it is important, is the present one where we compute the electric field a distance d from an infinite charged plane.

Let the plane have charge σ per unit area. We shall calculate $\mathbf{E}$ at a point P where P is a vertical distance d from the plane.

Select a circular disc of area A whose centre meets a perpendicular from the point P. Then, on this disk, erect a cylinder, with base of area A, which extends a height d *above* the plane and also a height d *below* it. This closed cylinder is chosen to be the surface S for Gauss's theorem. We note that left–right symmetry of the infinite plane forbids the field $\mathbf{E}$ from pointing in any direction other than *perpendicular* to the plane. This means that the integral over the curved part of S drops out since $\mathbf{E}$ is perpendicular to $\mathbf{dS}$ there. More precisely we find that

$$\begin{aligned}\int_S \mathbf{E}\cdot\mathbf{dS} &= \int_{\text{two discs}} \mathbf{E}\cdot\mathbf{dS}\\ &= \int_{\text{two discs}} |\mathbf{E}||\mathbf{dS}|\\ &= |\mathbf{E}|\int_{\text{two discs}} |\mathbf{dS}|\\ &= |\mathbf{E}|2A\end{aligned} \tag{2.36}$$

Hence we have

$$|\mathbf{E}|2A = \frac{Q}{\epsilon_0} \tag{2.37}$$

where Q is the charge on the disc of area A. So, if we let σ be the density of charge per unit area on the plates, then we deduce at once that

$$\begin{aligned}|\mathbf{E}|2A &= \frac{\sigma A}{\epsilon_0}\\ \Rightarrow |\mathbf{E}| &= \frac{\sigma}{2\epsilon_0}\\ \Rightarrow \mathbf{E} &= \frac{\sigma}{2\epsilon_0}\mathbf{n}\end{aligned} \tag{2.38}$$

where $\mathbf{n}$ is a unit vector perpendicular to the plate.

We draw attention to the fact that $\mathbf{E}$ has been found to be *independent* of the distance d that the point P is from the plate. This artificial result is only because we have taken an infinite rather than a finite plate; nevertheless our result is still numerically reasonable for finite plates with P subject to the following restrictions: P is opposite the plate, not near any of its edges but a horizontal distance h away from the nearest edge with

$$d/h << 1 \tag{2.39}$$

CHAPTER III

Charges in motion: electric currents

§ 1. Electric currents and resistance

WE are now ready to depart from the realm of electrostatics and to consider *moving charges*. Among other things this will allow us to discuss electric currents for the first time and we shall do this now.

When an electric current moves in a conductor such as a copper wire, for example, there is one conduction electron per copper atom and this large number of electrons makes it useful to define a vector **J** known as the *current density* cf. Fig. 1. below. So we now have the following definition.

Definition (Current density **J**) *The current density* **J** *is a vector whose direction coincides with that of the velocity vector* **v** *of the conduction electrons in the conductor. Its magnitude is given by the amount of charge crossing a unit area within the conductor per unit time.*

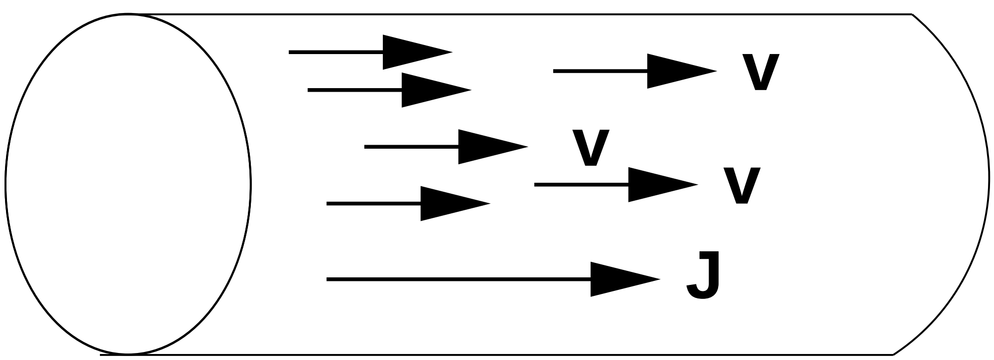

Fig. 1: Inside a conductor: electrons and the current density

We can easily find an expression for **J** and we now proceed to do just that: Let there be N electrons per unit volume inside the conductor then, in unit time, the electrons that cross a unit area travel a distance $|\mathbf{v}|$ (since they have velocity $\mathbf{v}$). Thus they trace out a cylinder of length $|\mathbf{v}|$ and base of unit area. The volume of this cylinder is therefore just

$$|\mathbf{v}| \times \mathbf{1} = |\mathbf{v}| \tag{3.1}$$

Hence the number of (conduction) electrons in this cylinder is precisely

$$N \times |\mathbf{v}| = N|\mathbf{v}| \tag{3.2}$$

and this means that the total *charge* in this cylinder is got by multiplying by e, where e is the electric charge; so this charge is

$$eN|\mathbf{v}| \tag{3.3}$$

But this number is, by the definition of **J** above, equal to the magnitude of **J** so we have deduced that

$$|\mathbf{J}| = eN|\mathbf{v}| \tag{3.4}$$

Finally the direction of **J** coincides with that of **v** so that the completed expression for the current density **J** is

$$\mathbf{J} = eN\mathbf{v} \tag{3.5}$$

The electron velocity vector **v** is usually referred to as the *drift velocity* of the electrons; this is because in most situations it has a rather small magnitude, we shall demonstrate this shortly in an example below.

Now the *electric current* I through any surface S is defined as follows:

Definition (Electric current I) *The electric current I through any surface S is defined to be the charge Q passing through S per unit time, i.e.*

$$I = \frac{dQ}{dt} \tag{3.6}$$

Also I is measured in amps.

Now if we take S to be the total cross section of a conductor, such as a copper wire, we see that I and **J** are related by integration over S giving us the equation

$$I = \int_S \mathbf{J} \cdot \mathbf{dS} \tag{3.7}$$

This brings us to the point where we can examine some of the details of the passage of current through a piece of conducting wire.

Example *The drift of electrons through a uniform straight copper wire*

Suppose that we pass a current of I amps through a uniform copper wire of cross sectional area A. Applying what we have just learned we write

$$\begin{aligned} I &= \int_S \mathbf{J} \cdot \mathbf{dS} \\ &= \int_S Ne\mathbf{v} \cdot \mathbf{dS} \end{aligned} \tag{3.8}$$

But in a straight wire **v** will be parallel to **dS** giving

$$\mathbf{v} \cdot \mathbf{dS} = |\mathbf{v}||\mathbf{dS}| \tag{3.9}$$

and we obtain the result that

$$\begin{aligned} I &= \int_S Ne|\mathbf{v}||\mathbf{dS}| \\ &= Ne|\mathbf{v}| \int_S |dS|, \quad \text{since } N,\ e \text{ and } |\mathbf{v}| \text{ are all constants} \\ \Rightarrow I &= Ne|\mathbf{v}|A, \quad \text{where } A \text{ is the cross sectional area of the wire} \end{aligned} \tag{3.10}$$

Now we can put in some typical numbers and see how small the drift velocity $|\mathbf{v}|$ actually is. Let

$$I = 1.5 \quad \text{amps},\ A = 1 \text{ square mm} = 10^{-6} m^2 \tag{3.11}$$

Further we know that, for copper,

$$N = 8 \times 10^{28}, \quad \text{electrons per } m^3 \tag{3.12}$$

and the charge e on an electron is given by

$$e = 1.6 \times 10^{-19} coul. \tag{3.13}$$

Hence since we can deduce from 3.10 that

$$|\mathbf{v}| = \frac{I}{NeA} \tag{3.14}$$

we find that

$$\begin{aligned} |\mathbf{v}| &= \frac{1.5}{8 \times 10^{28} \times 1.6 \times 10^{-19} \times 10^{-6}}, \quad m/sec \\ \Rightarrow |\mathbf{v}| &\simeq 10^{-4}, \quad m/sec \end{aligned} \tag{3.15}$$

which is indeed small justifying the name drift velocity for $|\mathbf{v}|$.

Continuing in our examination of the inner workings of a copper wire we now turn to the celebrated Ohm's law.

§ 2. Ohm's law

If a conductor has a potential V applied to it, causing a current I to flow, then this potential difference creates an internal electric field $\mathbf{E}$ where $\mathbf{E} = -\nabla V$. For most conductors this internal field $\mathbf{E}$ and the current density $\mathbf{J}$ are parallel. In other words, inside the conductor, one has [1]

$$\mathbf{J} = \sigma \mathbf{E}, \quad \sigma \text{ a constant} \tag{3.16}$$

This constant σ is called the *conductivity* of the conductor and its units of measurement are $ohm^{-1}\, m^{-1}$; for copper one has

$$\sigma = 5.9 \times 10^7\, ohm^{-1}\, m^{-1} \tag{3.17}$$

[1] This equation $\mathbf{J} = \sigma \mathbf{E}$ is sometimes referred to as Ohm's law as well as the more familiar equation $V = RI$; we shall see below that the former implies the latter so that there is some justification in such a nomenclature.

The *inverse* of σ is also used; it is denoted by ρ and is called the *resistivity* so that we can write

$$\rho = \frac{1}{\sigma} \tag{3.18}$$

Example *A wire of length L and cross section A*

Consider a wire of length L and cross sectional area A to which a potential difference V is applied. The potential V produces an internal electric field $\mathbf{E}$ and the two are related by

$$\begin{aligned} V &= \int_0^L \mathbf{E} \cdot \mathbf{dl} = |\mathbf{E}| \int_{\mathbf{0}}^{\mathbf{L}} |\mathbf{dl}|, \quad \text{since } \mathbf{E} \text{ is parallel to } \mathbf{dl} \\ \Rightarrow V &= |\mathbf{E}|L \end{aligned} \tag{3.19}$$

Also I is related to $\mathbf{J}$ by

$$\begin{aligned} I &= \int_S \mathbf{J} \cdot \mathbf{dS} \\ &= \sigma \int_S \mathbf{E} \cdot \mathbf{dS}, \quad \text{using 3.16} \\ \Rightarrow I &= \sigma |\mathbf{E}| A \end{aligned} \tag{3.20}$$

But since $V = |\mathbf{E}|L$ then we can write

$$\begin{aligned} I &= \frac{\sigma V A}{L} \\ \Rightarrow V &= \left(\frac{L}{\sigma A} \right) I \end{aligned} \tag{3.21}$$

hence we have deduced the familiar version of Ohm's law

$$\begin{aligned} V &= RI \\ \text{with} \quad R &= \frac{L}{\sigma A} \end{aligned} \tag{3.22}$$

We recognise R as the *resistance* of the material; its units of measurement are Ohms which are denoted by Ω. It useful to note that

$$R \propto \frac{1}{A} \tag{3.23}$$

but, by contrast,

$$R \propto L \tag{3.24}$$

It is useful to be aware of the conductivities of some of the more common substances in the world and we provide a table below.

Material	Conductivity	
Copper	5.9×10^7	
Gold	4.1×10^7	
Germanium	2.2	(semiconductor)
NaCl solution	23.0	
Glass	10^{-10}—10^{-14}	(insulator)
Quartz	1.3×10^{-18}	(piezo-electric effect)
Wood	10^{-8}—10^{-11}	

Conductivity table for substances $(293^0 K)$**.**

§ 3. The power dissipated in a wire

It is of great importance to be able to calculate the power dissipated by the passage of a current through a wire,

This is not difficult to do and one proceeds as follows: The force $\mathbf{F}$ on a charge q placed in an electric field $\mathbf{E}$ is given by

$$\mathbf{F} = q\mathbf{E} \tag{3.25}$$

If this force F moves this charge a distance $|\mathbf{dl}|$ in the direction $\mathbf{dl}$ then the work done is

$$\mathbf{F} \cdot \mathbf{dl} = q\mathbf{E} \cdot \mathbf{dl} \tag{3.26}$$

hence the total work done in moving the charge along a path Γ is given by

$$\begin{aligned} \int_\Gamma q\mathbf{E} \cdot \mathbf{dl} &= q \int_\Gamma \mathbf{E} \cdot \mathbf{dl} \\ &= -qV, \quad \text{since } V = -\int_\Gamma \mathbf{E} \cdot \mathbf{dl}, \quad \text{(cf. 1.29)} \end{aligned} \tag{3.27}$$

and we remind the reader that V is the voltage difference between the two ends of the path Γ. Next we apply this little piece of work to a current carrying wire.

Let a voltage difference V be applied to a wire of resistance R producing a current I. Then each charge q making up the current of the wire has an amount of work W done on it so that the total work done on the charges in the wire is W where

$$\begin{aligned} W &= \sum_{\text{charges}} -qV \\ &= -QV, \quad (\text{where } Q = \sum_{\text{charges}} q) \end{aligned} \tag{3.28}$$

Now the *sign* of the voltage *difference* V is arbitrary, since we have not said which end of the wire is which; so, for convenience, we shall change V to $-V$ and this means that the work done on all the charges in the wire is now

$$QV \tag{3.29}$$

But this work done comes from the internal energy in the voltage source—e.g. the chemical energy of a battery—so if U denotes the internal energy of the charges in the wire then we know that

$$U = QV \tag{3.30}$$

The rate of change of U with time is the energy consumed per unit time by the passage of the current—i.e. it is the power dissipated. If we denote the power dissipated by P then we have

$$\begin{aligned} P &= \frac{d}{dt}(QV) \\ &= \frac{dQ}{dt}V, \quad \text{assuming } V \text{ is constant} \\ &= IV, \quad \text{since } I = \frac{dQ}{dt} \\ \Rightarrow P &= VI \end{aligned} \tag{3.31}$$

But since Ohm's law says that $V = RI$ we can use Ohm's law to obtain three completely equivalent expressions for P and these are

$$\begin{aligned} P &= VI \\ P &= \frac{V^2}{R} \\ P &= I^2R \end{aligned} \tag{3.32}$$

CHAPTER IV

Magnetic fields

§ 1. Magnetic fields and Maxwell's second equation

THE forces between magnetic poles obey an inverse square law just as is the case for electric charges. This means that the magnetic field **B** obeys a flux law similar to that for electric fields **E**. Recall that for electric fields the inverse square law leads directly to Gauss's dielectric flux theorem which states that, for a closed surface S containing a total charge Q, the flux of **E** through S satisfies

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \frac{Q}{\epsilon_0} \tag{4.1}$$

Hence for magnetic fields the inverse square law says that, for a closed surface S containing a total *magnetic* charge Q_M, the flux of **B** through S satisfies

$$\int_S \mathbf{B} \cdot \mathbf{dS} = \frac{Q_M}{C} \tag{4.2}$$

where C is some constant which is the magnetic equivalent of ϵ_0. However *experimentally* it is found that *all magnetic charges occur in equal and opposite pairs.* This is often stated as the fact that no *magnetic monopoles* have ever been discovered. The conclusion that one draws from this is that the total amount of magnetic charge inside a surface is always exactly *zero* i.e. we *always* have

$$Q_M = 0 \tag{4.3}$$

But this means that

$$\int_S \mathbf{B} \cdot \mathbf{dS} = 0 \tag{4.4}$$

and so applying Gauss's divergence theorem to V, the volume contained inside S, we have

$$\begin{aligned} \int_S \mathbf{B} \cdot \mathbf{dS} &= \int_V \nabla \cdot \mathbf{B} \, dV \\ \Rightarrow \int_V \nabla \cdot \mathbf{B} \, dV &= 0 \\ \Rightarrow \nabla \cdot \mathbf{B} &= 0, \quad \text{since } V \text{ is arbitrary} \end{aligned} \tag{4.5}$$

This last result is very important as it is the second of Maxwell's four equations. Emphasising this we write

$$\nabla \cdot \mathbf{B} = 0, \quad \text{Maxwell's second equation} \tag{4.6}$$

Just to summarise, the two (of the four) Maxwell equations that we have derived so far are

Maxwell's first two equations

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}, \quad \text{Maxwell's first equation} \\ \nabla \cdot \mathbf{B} &= 0, \quad \text{Maxwell's second equation} \end{aligned} \tag{4.7}$$

We shall come to the last two in due course.

Another vital experimental fact is that which tells one how a moving charge is acted on by a magnetic field. This action is usually called the *Lorentz force law*. More formally we have [1]

The Lorentz force law shows how a magnetic field exerts a force on a moving charge

Lorentz force law *A charge q which moves with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ experiences a force $\mathbf{F}$, known as the Lorentz force, where*

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{4.8}$$

§ 2. Electric currents produce magnetic fields: Ampère's Theorem

A key result—due to Ampère—which helps one to measure and calculate the magnetic field $\mathbf{B}$ that is created when a current I passes through a wire is called Ampère's law (or Ampére's theorem) and its formal statement is;

Ampère's law or theorem *Let a current I pass through a wire thereby producing a magnetic field $\mathbf{B}$. Then if C is a closed curve we have the fact that*

$$\int_C \mathbf{B} \cdot \mathbf{dl} = \begin{cases} \mu_0 N I, & \text{if } C \text{ links } N \text{ times with the wire} \\ 0, & \text{otherwise} \end{cases} \tag{4.9}$$

Ampère's law is as useful for calculating magnetic fields as Gauss's dielectric flux theorem is for calculating electric fields. The most fruitful way to appreciate the importance of this result is to use it to obtain the magnetic field in a specific example. Let us now do this. Here is our first example.

Example *The magnetic field due to a long straight current carrying wire*

Let P be a point which is a perpendicular distance r from an infinite wire through which is passing current I; we want to calculate the magnetic field $\mathbf{B}$ at P. Since we wish to use Ampère's theorem we must first select a curve C around which to integrate the magnetic field $\mathbf{B}$; we choose C to be circle of radius $r = |\mathbf{r}|$ with its centre on the wire cf. Fig. 2.

[1] We shall use the Lorentz force law later when we derive Maxwell's third equation.

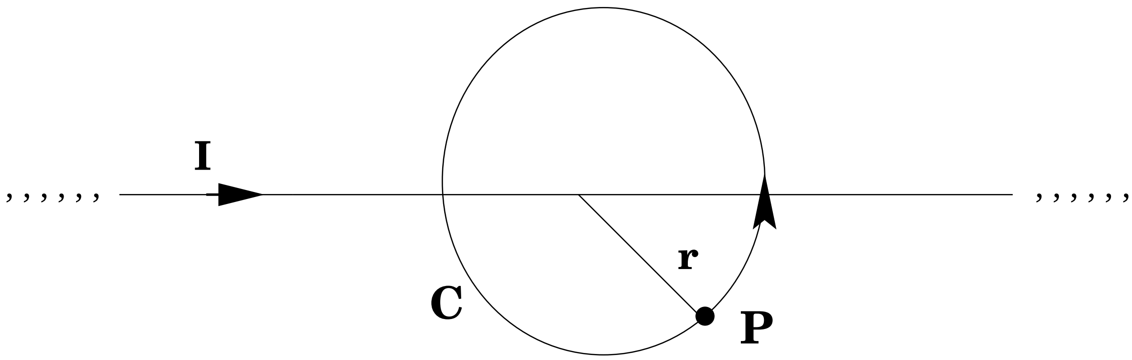

Fig. 2: The magnetic field produced by an infinite straight wire

Since this circle links the wire precisely once we have

$$\int_C \mathbf{B} \cdot \mathbf{dl} = \mu_0 I \tag{4.10}$$

Now we *assume* [2] that experiment has shown us that the magnetic lines of force are circles centred on the wire. This means that the **B** vectors are *tangential* to the curve C; but so are the **dl** vectors by their definition—i.e. **B** and **dl** are parallel. Hence

$$\mathbf{B} \cdot \mathbf{dl} = ||\mathbf{B}||\mathbf{dl}| \tag{4.11}$$

Now we can evaluate the integral for, using this parallelism, we have

$$\begin{aligned}\int_C \mathbf{B} \cdot \mathbf{dl} &= \int_C ||\mathbf{B}||\mathbf{dl}| \\ &= |\mathbf{B}| \int_{\mathbf{C}} |\mathbf{dl}|, \quad \text{since } |\mathbf{B}| \text{ is constant on } C \\ &= |\mathbf{B}| 2\pi r\end{aligned} \tag{4.12}$$

where we explain that **B** is constant on C since all points on C are the same perpendicular distance r from the wire; also we used the fact that $\int_C |dl|$ is just the total length of C—i.e. the circumference $2\pi r$ of the circle. Finally Ampère's law tells that the integral is equal to $\mu_0 I$ so we can say that

$$\begin{aligned}|\mathbf{B}| 2\pi r &= \mu_0 I \\ \Rightarrow |\mathbf{B}| &= \frac{\mu_0 I}{2\pi r}\end{aligned} \tag{4.13}$$

Thus, since we know that **B** is tangential to C, we let **e** denote a unit vector tangent to the curve C and the complete expression for the magnetic field **B** at P is now

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r} \mathbf{e} \tag{4.14}$$

[2] We shall redo this calculation without this assumption in the next section using the more powerful (but not always needed) Biot-Savart law.

We move on to another example, this one involves a *solenoid*.

Example *The magnetic field inside an infinite solenoid*

This time we pass a current I through an infinitely long solenoid—cf. Fig. 3 for a picture of a short piece of the solenoid viewed from a skew angle.

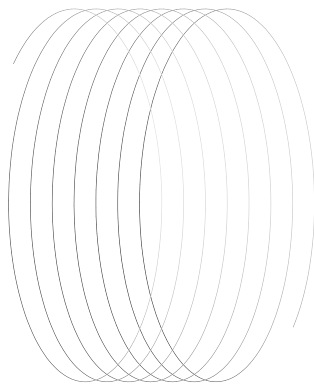

Fig. 3: A loosely wound solenoid

If we move round exactly perpendicular to the axis of the solenoid and stretch it out somewhat it will then look as shown in Fig. 4 below.

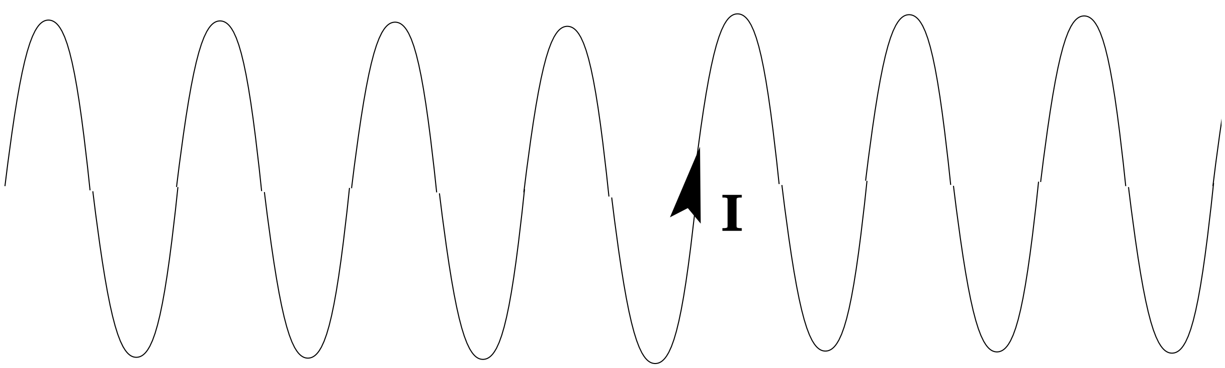

Fig. 4: A loosely wound solenoid carrying a current I

The current I produces a magnetic field **B**. We want an expression for the field **B** at the point P where P is a point on the axis of the solenoid. We are going to use Ampère's theorem and so must choose a curve C and then integrate **B** around C. We choose C to be the rectangle $EFGH$, cf. Fig. 5.

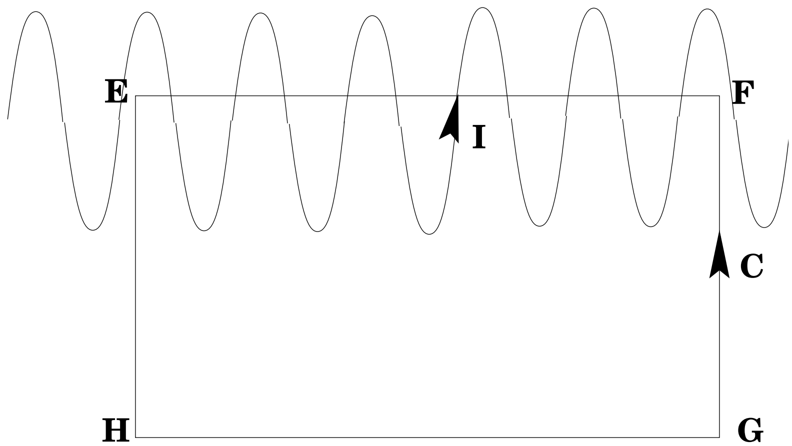

Fig. 5: The solenoid and the rectangular path C

Next we must specify *how tightly* the solenoid is wound and so we define the integer N by saying that N is the *number of turns per unit length* of the solenoid. In addition we wish to specify the *width* of the rectangle, i.e the length of the line EF; we shall denote the length of EF by L.

We note, now. that all this means that the line EF, having length L, passes through exactly

$$NL \tag{4.15}$$

turns of the solenoid. This in turn means that Ampère's theorem applied to the curve C gives the result that

$$\int_C \mathbf{B}\cdot\mathbf{dl} = \mu_0 NLI \tag{4.16}$$

since the rectangle is linked with all the NL turns passed through by the line EF.

The final short task that we have is to evaluate the integral $\int_C \mathbf{B}\cdot\mathbf{dl}$. The key to doing this is comprised of two observations and these are

(i) The integral $\int_C \mathbf{B}\cdot\mathbf{dl}$ is *independent* of the *length* FG of C; hence we may make this length as large as we like.

(ii) Axial symmetry of an infinitely long object, such as this solenoid, means that the magnetic field $\mathbf{B}$ must point along the axis of the solenoid. This then means that $\mathbf{B}$ is perpendicular to the $\mathbf{dl}$ vectors on the two vertical sides of C: namely the sides GF and EH. To use these observations we first decompose the integral into four pieces—one for each side of the rectangle—giving

$$\int_C \mathbf{B}\cdot\mathbf{dl} = \int_{HG} \mathbf{B}\cdot\mathbf{dl} + \int_{GF} \mathbf{B}\cdot\mathbf{dl} + \int_{FE} \mathbf{B}\cdot\mathbf{dl} + \int_{EH} \mathbf{B}\cdot\mathbf{dl} \tag{4.17}$$

Now, because of the perpendicularity mentioned in (ii) above, the integrals along the sides GF and EH vanish, so we have

$$\int_{GF} \mathbf{B}\cdot\mathbf{dl} = \int_{EH} \mathbf{B}\cdot\mathbf{dl} = 0 \tag{4.18}$$

Also, exploiting point (i) above, we make the length GF tend to infinity this make $\mathbf{B}$ tend to zero along HG so the integral for this side vanishes giving

$$\int_{HG} \mathbf{B}\cdot\mathbf{dl} = 0 \tag{4.19}$$

All that remains of the integral around C is the portion FE; but this portion is along the axis where $\mathbf{B}$ is parallel to the $\mathbf{dl}$ vectors. Hence we can immediately compute that

$$\int_{FE} \mathbf{B}\cdot\mathbf{dl} = \int_{FE} |\mathbf{B}||\mathbf{dl}| = |\mathbf{B}|\int_{FE} |\mathbf{dl}| = |\mathbf{B}|L \tag{4.20}$$

So the upshot of doing these four integrals, when combined with 4.16, is that

$$\begin{aligned} \int_C \mathbf{B}\cdot\mathbf{dl} &= |\mathbf{B}|L = \mu_0 NLI \\ \Rightarrow |\mathbf{B}| &= \mu_0 NI \end{aligned} \tag{4.21}$$

This gives us the magnitude of the field $\mathbf{B}$ inside the solenoid, but we already know that $\mathbf{B}$ point along the axis of the solenoid; hence, if $\mathbf{e}$ denotes a unit vector along the axis of the solenoid, our final result for $\mathbf{B}$ is that

$$\mathbf{B} = \mu_0 NI\mathbf{e} \tag{4.22}$$

§ 3. The Biot–Savart law

The Ampère law can sometimes not yield an easy route to the calculation of the magnetic field produced by a current carrying wire. When this is so there is a more powerful result that one can have recourse to; this result is called the Biot–Savart law and it is the following statement.

Biot-Savart law *Let a current I pass through a wire thereby producing a magnetic field* $\mathbf{B}$*. Then if* $\mathbf{dl}$ *is an element of length along the wire, located at* $\mathbf{r}'$*, it produces a field* $\mathbf{dB}$ *at the point* $\mathbf{r}$ *where*

$$\mathbf{dB} = \frac{\mu_0 I}{4\pi}\frac{\mathbf{dl}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{4.23}$$

The best way to understand this law is to move straight on to an example. We shall work through two examples as illustrations of the Biot–Savart law; as our first example we shall redo the calculation of the field $\mathbf{B}$ due to a straight wire that we computed using Ampère's law.

Example *The magnetic field due to a long straight current carrying wire redone using the Biot–Savart law*

We shall take the wire to coincide with the x'-axis, cf. Fig. 6 (note that we have to call this axis the x'-axis rather than the x-axis because we have already used the variable x in the expression for the vector $\mathbf{r}$ which we recall is given by $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$), so that

$$\mathbf{dl} = dx'\mathbf{i} \tag{4.24}$$

In addition we shall choose (without any loss of generality) the point $\mathbf{r}$ to lie in the $x - z$ plane. With these notational conventions we have

$$\mathbf{r}' = x'\mathbf{i}, \quad \mathbf{r} = x\mathbf{i} + z\mathbf{k} \tag{4.25}$$

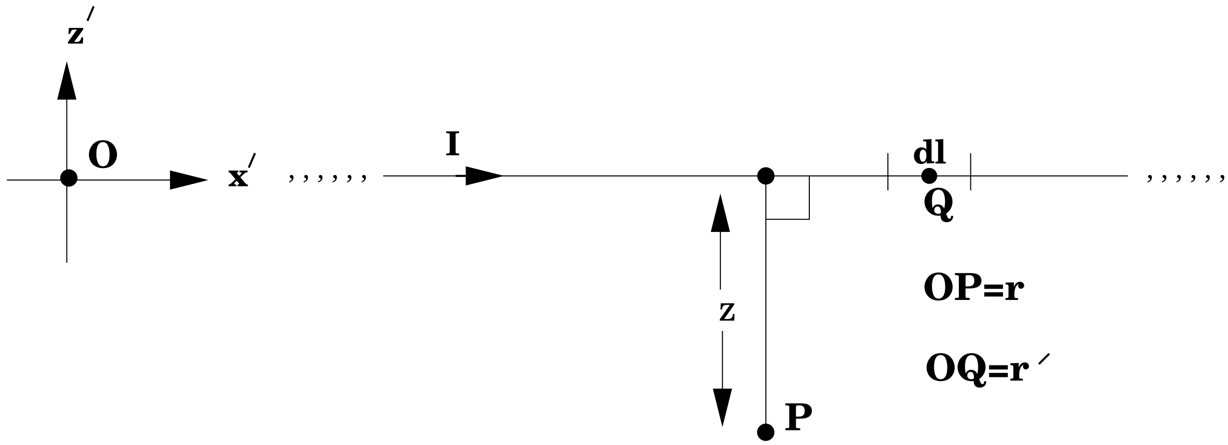

Fig. 6: The straight wire for the Biot-Savart calculation

It is now straightforward to calculate that

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{(x - x')^2 + z^2} \tag{4.26}$$

and that

$$\begin{aligned} \mathbf{dl} \times (\mathbf{r} - \mathbf{r}') &= dx'\mathbf{i} \times \{(x - x')\mathbf{i} + z\mathbf{k}\} \\ &= z dx'\mathbf{i} \times \mathbf{k} = -z dx'\mathbf{j} \end{aligned} \tag{4.27}$$

So now we have

$$\mathbf{dB}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi} \frac{z dx'}{\{(x - x')^2 + z^2\}^{3/2}} \mathbf{j} \tag{4.28}$$

The last step in the calculation is to integrate **dB**: we have

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0 I z}{4\pi} \mathbf{j} \int_{-\infty}^{\infty} \frac{dx'}{\{(x - x')^2 + z^2\}^{3/2}} \tag{4.29}$$

Evaluating the integral is routine enough: we make the substitution

$$\begin{aligned} x - x' &= z \tan(\theta) \\ \Rightarrow dx' &= -z \sec^2(\theta) d\theta \end{aligned} \tag{4.30}$$

and then we find that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dx'}{\{(x - x')^2 + z^2\}^{3/2}} &= -\int_{-\pi/2}^{\pi/2} d\theta \, \frac{z \sec^2(\theta)}{z^3 \sec^3(\theta)} \\ &= -\frac{1}{z^2} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sec(\theta)} = -\frac{1}{z^2} \int_{-\pi/2}^{\pi/2} d\theta \, \cos(\theta) \\ &= -\frac{2}{z^2} \end{aligned} \tag{4.31}$$

The resulting expression for **B** is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi z} \mathbf{j} \tag{4.32}$$

and this is in complete agreement with the expression 4.14 obtained using Ampère's law except that in 4.14 the variable z was denoted by r and the unit vector $\mathbf{e}$ is here identified as $\mathbf{j}$.

Example *The magnetic field at the centre of a circular loop carrying a current I*

Let a current I pass through a closed wire which is bent into a circular shape, the circle having a radius R. We want to calculate the magnetic field $\mathbf{B}$ produced at the centre of the circle.

The basic Biot–Savart expression for the field due to the element of wire $\mathbf{dl}$ is

$$\mathbf{dB} = \frac{\mu_0 I}{4\pi}\frac{\mathbf{dl}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{4.33}$$

Now we choose the wire to lie in the $x - z$ plane so that the *centre* of the circle coincides with the *origin* of the coordinate system. This has the great simplification that the point $\mathbf{r}$ at which we want to calculate $\mathbf{B}$ is the zero vector—i.e. we have

$$\mathbf{r} = \mathbf{0} \tag{4.34}$$

This means that $\mathbf{dB}$ is now of the form

$$\mathbf{dB} = -\frac{\mu_0 I}{4\pi}\frac{\mathbf{dl}\times\mathbf{r}'}{|\mathbf{r}'|^3} \tag{4.35}$$

Now the vector $\mathbf{r}'$ lies on the circle of radius R so it is given by the formula

$$\mathbf{r}' = R\cos(\theta)\mathbf{i} + R\sin(\theta)\mathbf{k} \tag{4.36}$$

from which we see immediately that

$$|\mathbf{r}'| = R \tag{4.37}$$

Hence

$$\mathbf{dB} = -\frac{\mu_0 I}{4\pi}\frac{\mathbf{dl}\times(R\cos(\theta)\mathbf{i} + R\sin(\theta)\mathbf{k})}{R^3} \tag{4.38}$$

Now the derivative

$$\frac{\mathbf{dr}'}{d\theta} \tag{4.39}$$

has to be tangential to the circle at the point $\mathbf{r}'$ so the vector $\mathbf{dl}$, which is also tangent at the same point, but has length $Rd\theta$, is given by

$$\begin{aligned}\mathbf{dl} &= \frac{\mathbf{dr}'}{d\theta}d\theta \\ &= (-R\sin(\theta)\mathbf{i} + R\cos(\theta)\mathbf{k})d\theta \\ &= R(-\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{k})d\theta\end{aligned} \tag{4.40}$$

So, putting all this together, we have

$$\mathbf{dB} = -\frac{\mu_0 I}{4\pi}\frac{R\,(-\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{k})\,d\theta\times(R\cos(\theta)\mathbf{i} + R\sin(\theta)\mathbf{k})}{R^3} \tag{4.41}$$

Tidying up, and doing the cross products, we find that

$$\begin{aligned}\mathbf{dB} &= -\frac{\mu_0 I}{4\pi R}\left(\sin^2(\theta)\mathbf{j} + \cos^2(\theta)\mathbf{j}\right) d\theta \\ &= -\frac{\mu_0 I}{4\pi R}\left(\sin^2(\theta) + \cos^2(\theta)\right)\mathbf{j} d\theta \\ &= -\frac{\mu_0 I}{4\pi R}\mathbf{j} d\theta\end{aligned} \tag{4.42}$$

It is now a trivial matter to integrate and obtain

$$\begin{aligned}\mathbf{B} &= \int \mathbf{dB} \\ &= -\int_0^{2\pi} \frac{\mu_0 I}{4\pi R}\mathbf{j} d\theta \\ &= 2\pi \\ \Rightarrow \mathbf{B} &= -\frac{\mu_0 I}{2R}\mathbf{j}\end{aligned} \tag{4.43}$$

and so our calculation is complete; note, before leaving this example, that $|\mathbf{B}| \propto 1/R$.

An equation for $\nabla \times \mathbf{B}$

We are now in a position to derive a useful equation for $\nabla \times \mathbf{B}$ which will turn out later to be a stepping stone to Maxwell's fourth equation. The equation we are after is

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{4.44}$$

To derive it we start with a curve C encircling a current carrying wire just once; this means that Ampère's law says that

$$\int_C \mathbf{B} \cdot \mathbf{dl} = \mu_0 I \tag{4.45}$$

But, if S is the surface interior to C, and $\mathbf{J}$ is the current density, then I is given by

$$I = \int_S \mathbf{J} \cdot \mathbf{dS} \tag{4.46}$$

Inserting this into Ampère's law gives

$$\int_C \mathbf{B} \cdot \mathbf{dl} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{dS} \tag{4.47}$$

where I is the current carried by the wire. But applying Stokes' theorem to the $\mathbf{B}$ integral we then obtain

$$\int_S \nabla \times \mathbf{B} \cdot \mathbf{dS} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{dS} \tag{4.48}$$

Finally, since the surface S is arbitrary—except that it must be cut by the wire—we find that the integrands are equal, i.e. we have the desired result

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{4.49}$$

CHAPTER V

Maxwell's third and fourth equations

§ 1. Electromagnetic induction and Maxwell's third equation

WE now begin a discussion which will end with the derivation of Maxwell's third equation. As a prerequisite we recall the *Lorentz force law* quoted on p. 22 which told us us how a magnetic field acts on a moving charge. Here, once again, is the formal statement of the law:

Lorentz force law *A charge q which moves with velocity* $\mathbf{v}$ *in a magnetic field* $\mathbf{B}$ *experiences a force* $\mathbf{F}$*, known as the Lorentz force, where*

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{5.1}$$

We turn now to electromagnetic induction. The great experimentalist Faraday discovered that if one takes a closed loop of wire through which *no current is passing* then, if one *moves the loop* in a magnetic field $\mathbf{B}$, a current I can be induced.

Let the loop of wire enclose a surface S, then as the loop moves the magnetic flux $\int_S \mathbf{B} \cdot \mathbf{dS}$ changes with time. Faraday's very ingenious and careful experiments established that the voltage [1] $\mathcal{E}$, producing this current is related to the rate of change of the magnetic flux Φ by the equation

$$\begin{aligned} \mathcal{E} &= -\frac{\partial}{\partial t}\int_S \mathbf{B} \cdot \mathbf{dS} \\ \text{i.e.} \qquad \mathcal{E} &= -\frac{\partial \Phi}{\partial t} \end{aligned} \tag{5.2}$$

We shall now use the Lorentz force to rederive Faraday's result and to derive Maxwell's third equation which happens to be

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5.3}$$

Let us take a rectangular loop $ABCD$ placed in a constant magnetic field $\mathbf{B}$ which is *perpendicular* to the plane of the rectangle so that is parallel to the $\mathbf{dS}$ vector. In addition

[1] $\mathcal{E}$ is also often called an *emf*, this stands for the term *electromotive force*

we want the side AB to be moveable and be able to slide along at a constant speed v cf. Fig. 7.

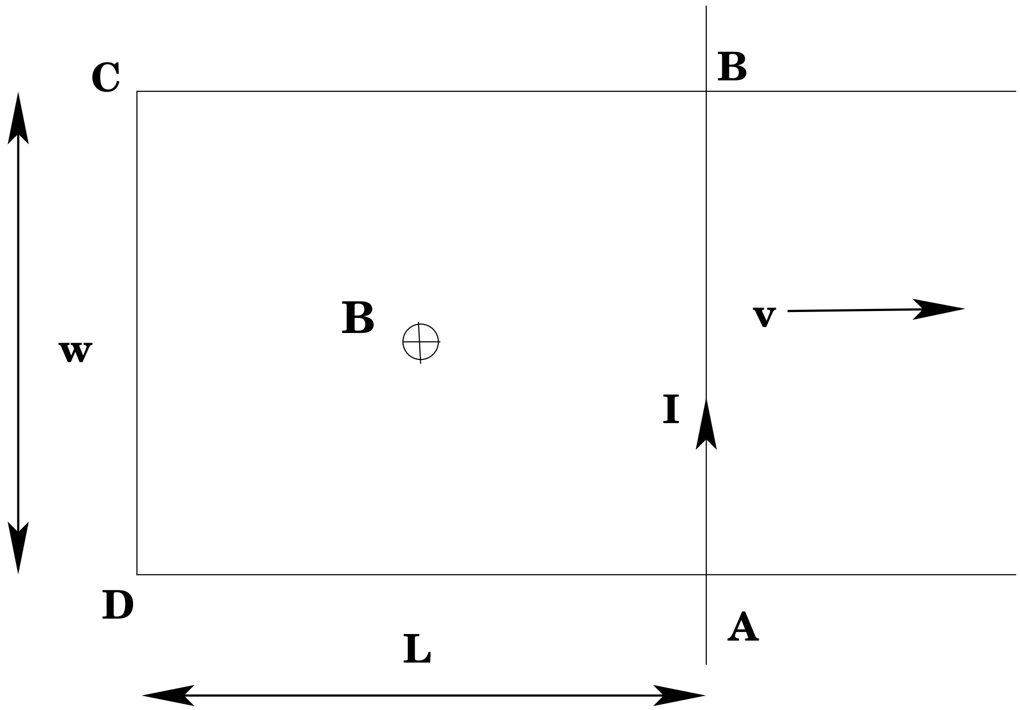

Fig. 7: The rectangular loop with the moving side

Now, since $\mathbf{B}$ is constant and perpendicular to the rectangle, the flux Φ through this loop is just $|\mathbf{B}|$ times the area of the loop, so that we have

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS} = |\mathbf{B}|wL \tag{5.4}$$

the area of the loop being wL. Further if we differentiate Φ with respect to t, and notice that

$$\frac{dL}{dt} = |\mathbf{v}| \tag{5.5}$$

since the side AB is moving with velocity $\mathbf{v}$, then we see at once that

$$\frac{\partial \Phi}{\partial t} = |\mathbf{B}|w|\mathbf{v}| \tag{5.6}$$

Next note that the electrons in the moving side AB are moving (with velocity $\mathbf{v}$) in a magnetic field and so are subject to the Lorentz force law. They receive, therefore, a force

$$\mathbf{F} = e\mathbf{v} \times \mathbf{B} \tag{5.7}$$

But this force is the same as if they were subject to an electric field $\mathbf{E}$ given by the expression

$$\mathbf{E} = \mathbf{v} \times \mathbf{B} \tag{5.8}$$

and such an electric field would be produced by applying a potential difference $\mathcal{E}$ across the ends of AB where[2]

$$\mathcal{E} = \int_{AB} \mathbf{v} \times \mathbf{B} \cdot \mathbf{dl} \tag{5.9}$$

However $\mathbf{v}$, $\mathbf{B}$ and $\mathbf{dl}$ are all constant and mutually perpendicular so we have immediately the result that[3]

$$\begin{aligned} \int_{AB} \mathbf{v} \times \mathbf{B} \cdot \mathbf{dl} &= -|\mathbf{v}||\mathbf{B}|w \\ \text{i.e.} \qquad \mathcal{E} &= -|\mathbf{v}||\mathbf{B}|w \end{aligned} \tag{5.10}$$

Now if we compare equations 5.6 and 5.10 for Φ and $\mathcal{E}$ respectively we find that we do indeed have

$$\mathcal{E} = -\frac{\partial \Phi}{\partial t} \tag{5.11}$$

in agreement with Faraday's experimental law.

Since the other three sides of the rectangle do not move, we can change the expression for $\mathcal{E}$ to be an integral all the way around the rectangle and not just along the side AB; we shall do this and, denoting the rectangle $ABCD$ by just C, we have

$$\mathcal{E} = \int_C \mathbf{E} \cdot \mathbf{dl} \tag{5.12}$$

Maxwell's equation follows from being more formal and writing

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}, \qquad \text{and} \qquad \mathcal{E} = \int_C \mathbf{E} \cdot \mathbf{dl} \tag{5.13}$$

With this notation 5.11 becomes

$$\begin{aligned} \frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{dS} &= -\int_C \mathbf{E} \cdot \mathbf{dl} \\ \Rightarrow \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS} &= -\int_C \mathbf{E} \cdot \mathbf{dl} \\ \Rightarrow \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS} &= -\int_S \nabla \times \mathbf{E} \cdot \mathbf{dS}, \qquad \text{by Stokes' theorem} \\ \Rightarrow \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \qquad \text{since } S \text{ is arbitrary} \end{aligned} \tag{5.14}$$

and so we have the *third* of Maxwell's four equations. Stating it again for emphasis, it is the equation

[2] Note that the convention is to define $\mathcal{E} = \int_{AB} \mathbf{E} \cdot \mathbf{dl}$ rather than $\mathcal{E} = -\int_{AB} \mathbf{E} \cdot \mathbf{dl}$ so that $\mathcal{E}$ is actually the opposite sign to what we normally call a voltage.

[3] To get the signs on the RHS work out you have choose $\mathbf{B}$ 'perpendicular and upwards' to $ABCD$ and recall that $\mathbf{dl}$ points from A to B; this means that the direction of $\mathbf{E} = \mathbf{v} \times \mathbf{B}$ is *opposite* to that of $\mathbf{dl}$. Other choices can be made but the final result will be unaltered.

Maxwell's third equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5.15}$$

We turn at once to the derivation of the fourth and last Maxwell equation.

§ 2. Maxwell's fourth equation—the story of the displacement current

Maxwell's fourth equation involves the quantity $\nabla \times \mathbf{B}$; now we have already obtained an equation for $\nabla \times \mathbf{B}$ namely

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{5.16}$$

however we shall now see that this equation is incomplete. It is the corrected, or completed, form of this equation that we are after.

Let us see what is wrong with

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{5.17}$$

First take the divergence of both sides yielding

$$\nabla \cdot \nabla \times \mathbf{B} = \mu_0 \nabla \cdot \mathbf{J} \tag{5.18}$$

But $\nabla \cdot \nabla \times \mathbf{B} = 0$ since $\nabla \cdot \nabla \times \mathbf{A} = 0$ for *any* $\mathbf{A}$. Hence we have deduced that

$$\nabla \cdot \mathbf{J} = 0 \tag{5.19}$$

Unfortunately this deduction is a *disaster*, and is definitely mistaken, since it contradicts the extremely well established experimental fact that charge is conserved. We now have to put this right.

Suppose, then, that the density of charge inside an arbitrary volume V is ρ and that the charges are inside V are not static but in motion giving rise to a current density $\mathbf{J}$. Now some charges may flow out across the surface S of V and escape thus reducing the total charge Q of V. The corresponding *decrease* in Q per unit time is therefore

$$-\frac{dQ}{dt} \tag{5.20}$$

This outward flow is a current I across the surface S and we know that

$$I = \int_S \mathbf{J} \cdot \mathbf{dS} \tag{5.21}$$

But I is also a measured in units of charge per unit time and, *since charge is conserved*, this current must be equal to the decrease in charge of V. In other words we must have

$$-\frac{dQ}{dt} = \int_S \mathbf{J} \cdot \mathbf{dS} \tag{5.22}$$

But we can express Q in terms of the charge density ρ by writing

$$Q = \int_V \rho dV \tag{5.23}$$

so we have

$$\begin{aligned} -\frac{\partial}{\partial t}\int_V \rho\, dV &= \int_S \mathbf{J}\cdot\mathbf{dS} \\ \Rightarrow -\int_V \frac{\partial \rho}{\partial t}\, dV &= \int_S \mathbf{J}\cdot\mathbf{dS} \end{aligned} \tag{5.24}$$

But Gauss's divergence theorem applied to $\mathbf{J}$ says that

$$\int_S \mathbf{J}\cdot\mathbf{dS} = \int_V \nabla\cdot\mathbf{J}\, dV \tag{5.25}$$

and so we find that

$$\begin{aligned} -\int_V \frac{\partial \rho}{\partial t}\, dV &= \int_V \nabla\cdot\mathbf{J}\, dV \\ \Rightarrow \int_V \left\{\nabla\cdot\mathbf{J} + \frac{\partial \rho}{\partial t}\right\} dV &= 0 \\ \Rightarrow \nabla\cdot\mathbf{J} + \frac{\partial \rho}{\partial t} &= 0, \quad \text{since } V \text{ is arbitrary} \end{aligned} \tag{5.26}$$

and this equation expresses the conservation of charge. So we see that, rather than having $\nabla\cdot\mathbf{J} = 0$ we have

$$\nabla\cdot\mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{5.27}$$

Our strategy now is to add a term to our equation for $\nabla\times\mathbf{B}$ and to try and derive the form of this term from what we know already. To this end we denote this added term by $\mathbf{J_D}$ and call it (following the common practice) the *displacement current*; the equation for $\nabla\times\mathbf{B}$ then becomes

$$\nabla\times\mathbf{B} = \mu_0\,\mathbf{J} + \mathbf{J_D} \tag{5.28}$$

Proceeding as before we take the divergence of both sides giving

$$\begin{aligned} \nabla\cdot\nabla\times\mathbf{B} &= \mu_0\nabla\cdot\mathbf{J} + \nabla\cdot\mathbf{J_D} \\ \Rightarrow 0 &= \mu_0\,\nabla\cdot\mathbf{J} + \nabla\cdot\mathbf{J_D} \\ \Rightarrow 0 &= -\mu_0\,\frac{\partial \rho}{\partial t} + \nabla\cdot\mathbf{J_D}. \quad \text{using 5.27} \end{aligned} \tag{5.29}$$

Here we take time off to point out that experimentally [4] it is found that the constant μ_0 is related to the constant ϵ_0; in fact one knows that

$$\mu_0 = \frac{1}{\epsilon_0 c^2} \tag{5.30}$$

[4] For the record this was work done in 1856 by Weber and Kohlrausch—cf. Weber, W., Kohlrausch, R., *Über die Elektrizitätsmenge, welche bei galvanischen Strömen durch den Querschnitt der Kette fließt*, Poggendorffs Annalen, **99**, 10–25, (1856)—this result of course strongly suggests that light has something to do with electromagnetic phenomena; however the velocity of light was not known all that well in 1856 though it had become much more accurately determined by the year 1864: the year when Maxwell did his famous work, cf. below.

where c is the velocity of light in vacuo. In any case we now have an equation for the displacement current $\mathbf{J_D}$, it is simply that

$$\nabla \cdot \mathbf{J_D} = \frac{1}{\epsilon_0 c^2}\frac{\partial \rho}{\partial t} \tag{5.31}$$

Now we integrate both sides of this equation over the volume V giving

$$\int_V \nabla \cdot \mathbf{J_D}\, dV = \frac{1}{\epsilon_0 c^2}\int_V \frac{\partial \rho}{\partial t}\, dV \tag{5.32}$$

But we know from Gauss's divergence theorem that

$$\int_V \nabla \cdot \mathbf{J_D}\, dV = \int_S \mathbf{J_D} \cdot \mathbf{dS} \tag{5.33}$$

and from Maxwell's first equation that

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\ \Rightarrow \rho &= \epsilon_0 \nabla \cdot \mathbf{E} \end{aligned} \tag{5.34}$$

Using both these facts we get

$$\begin{aligned} \int_S \mathbf{J_D} \cdot \mathbf{dS} &= \frac{1}{\epsilon_0 c^2}\int_V \epsilon_0 \frac{\partial(\nabla \cdot \mathbf{E})}{\partial t}\, dV \\ \Rightarrow \int_S \mathbf{J_D} \cdot \mathbf{dS} &= \frac{1}{c^2}\frac{\partial}{\partial t}\int_V \nabla \cdot \mathbf{E}\, dV \\ \Rightarrow \int_S \mathbf{J_D} \cdot \mathbf{dS} &= \frac{1}{c^2}\frac{\partial}{\partial t}\int_S \mathbf{E} \cdot \mathbf{dS}, \quad \text{(Gauss's divergence theorem)} \\ \Rightarrow \int_S \mathbf{J_D} \cdot \mathbf{dS} &= \int_S \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{dS} \end{aligned} \tag{5.35}$$

Then, as usual, since S is arbitrary we conclude that

$$J_D = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{5.36}$$

and we have successfully found the displacement current $\mathbf{J_D}$. This means that the correct equation for $\nabla \times \mathbf{B}$, which is *Maxwell's fourth equation* is

Maxwell's fourth equation

$$\nabla \times \mathbf{B} = \frac{1}{\epsilon_0 c^2}\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{5.37}$$

So we now have all four of Maxwell's equations.

In the next section we shall make some comments on each equation and examine the displacement current term a little more closely.

§ 3. Maxwell's four equations

The four Maxwell equations should now be viewed together and some thought given to how they were derived. To this end the four equations are listed below with each accompanied by a short relevant comment relating to their origins.

Maxwell's celebrated four equations

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}, \qquad \text{(Coulomb's law)} \\ \nabla \cdot \mathbf{B} &= 0, \qquad \text{(absence of magnetic monopoles)} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \qquad \text{(electromagnetic induction)} \\ \nabla \times \mathbf{B} &= \frac{1}{\epsilon_0 c^2}\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \qquad \text{(displacement current)} \end{aligned} \tag{5.38}$$

Example *The displacement current term in action*

The displacement current's rôle in electromagnetic phenomena can be quite subtle; because this is so we now present an example of the working of this term

$$\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{5.39}$$

We take a spherically symmetric situation. Let us have a radioactive β-source in the centre of a containing sphere. This source simply serves as a source of electrons which, by the spherical symmetry of the situation, are emitted outwardly along the radial directions giving a radial current density **J** cf. Fig. 8.

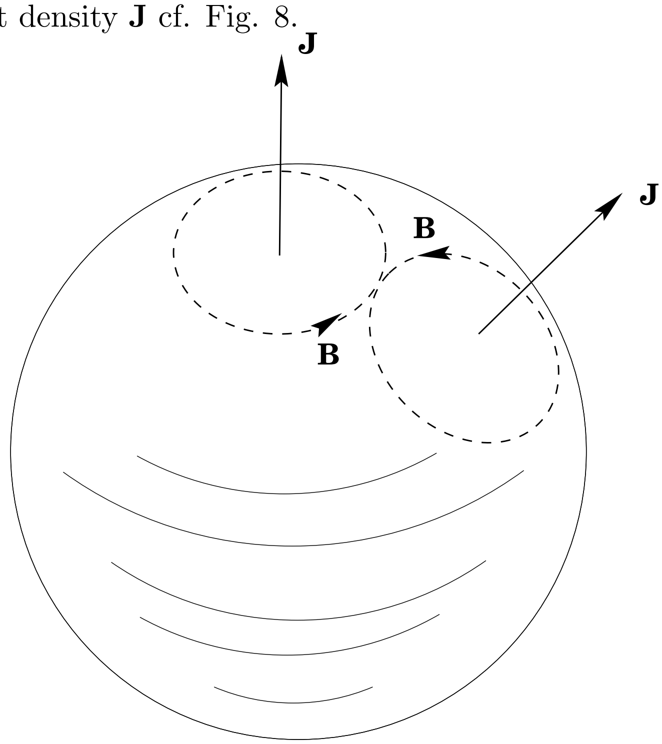

Fig. 8: The β-source with its radial current density **J**.

In Fig. 8 the dotted lines show the magnetic lines of force produced by each vector **J**. Note carefully that where the a pair of such lines touch the magnetic field the will be cancellation between the two causing the magnetic field $\mathbf{B}$ to be zero there. Since such a point of intersection could be anywhere on the sphere this suggests that $\mathbf{B}$ should be zero everywhere on the sphere. This is actually the case but we shall not prove it we shall just show that

$$\nabla \times \mathbf{B} = \mathbf{0} \tag{5.40}$$

which is, of course, implied by $\mathbf{B} = \mathbf{0}$.

The point is that $\nabla \times \mathbf{B}$ is the LHS of Maxwell's equation and so if it vanishes there must be a *cancellation* between the two terms on the RHS. In other words we will have

$$\begin{aligned} \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} &= -\frac{1}{\epsilon_0 c^2}\mathbf{J} \\ \Rightarrow \frac{\partial \mathbf{E}}{\partial t} &= -\frac{\mathbf{J}}{\epsilon_0} \end{aligned} \tag{5.41}$$

so that the displacement current term cannot be zero since $\mathbf{J}$ is, by construction, non-zero. We proceed to the calculation.

Let the sphere have radius r so and let it contain a total charge $Q(t)$ at time t. The *outward* electrical current I from the radioactive decay is given by

$$I = -\frac{\partial Q(t)}{\partial t} \tag{5.42}$$

But

$$I = \int_S \mathbf{J} \cdot \mathbf{dS} = 4\pi r^2 |\mathbf{J}| \tag{5.43}$$

so we have

$$-\frac{\partial Q(t)}{\partial t} = 4\pi r^2 |\mathbf{J}| \tag{5.44}$$

However, for the electric field $\mathbf{E}$, spherical symmetry says that $\mathbf{E}$ is the same as the field produced by a point charge at the centre of the sphere so we have

$$\begin{aligned} \mathbf{E} &= \frac{Q(t)}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2} \\ \Rightarrow Q(t) &= 4\pi\epsilon_0 |\mathbf{E}| r^2 \end{aligned} \tag{5.45}$$

Hence combining eqs. 5.44 and 5.45 we find that

$$\begin{aligned} \frac{\partial\left(4\pi\epsilon_0 r^2 |\mathbf{E}|\right)}{\partial t} &= -4\pi r^2 |\mathbf{J}| \\ \Rightarrow \frac{\partial |\mathbf{E}|}{\partial t} &= -\frac{|\mathbf{J}|}{\epsilon_0} \\ \Rightarrow \frac{\partial \mathbf{E}}{\partial t} &= -\frac{\mathbf{J}}{\epsilon_0}, \qquad \text{since } \mathbf{E} = |\mathbf{E}|\,\hat{\mathbf{r}} \text{ and } \mathbf{J} = |\mathbf{J}|\,\hat{\mathbf{r}} \end{aligned} \tag{5.46}$$

Hence we have verified that

$$\nabla \times \mathbf{B} = \mathbf{0} \tag{5.47}$$

which is what we wanted to do.

CHAPTER VI

Electromagnetic waves

§ 1. Wave solutions to Maxwell's equations

It is now time to establish the celebrated result that both **E** and **B** satisfy wave equations with the wave velocity being c—the velocity of light. In fact this is a simple consequence of Maxwell's equations so the difficult thing was to derive Maxwell's equations in the first place.

We start with the wave equation for the electric field **E**. First we set ρ and **J** to zero in Maxwell's equations giving us Maxwell's equations in vacuo, also called Maxwell's equations in free space: these are the four equations

Maxwell's equations in a vacuum: they describe electromagnetic radiation

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \end{aligned} \tag{6.1}$$

Taking the curl of Maxwell's equation for $\nabla \times \mathbf{E}$ yields

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \left(\frac{\partial \mathbf{B}}{\partial t}\right) \tag{6.2}$$

Now we point out that there is a vector identity which states that, for any **A**,

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{6.3}$$

Applied to **E** this gives, if we also use the fact that $\nabla \cdot \mathbf{E} = 0$,

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E} \tag{6.4}$$

But

$$\begin{aligned} \nabla \times \left(\frac{\partial \mathbf{B}}{\partial t}\right) &= \frac{\partial (\nabla \times \mathbf{B})}{\partial t} \\ &= \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad \text{on using Maxwell's fourth equation} \end{aligned} \tag{6.5}$$

So we have now deduced that

$$\begin{aligned} -\nabla^2 \mathbf{E} &= -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \\ \Rightarrow \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E} &= 0 \end{aligned} \tag{6.6}$$

But this latter is a wave equation and shows that $\mathbf{E}$ travels as a wave with velocity c where c is the velocity of light.

In an exactly similar fashion we can take the curl of Maxwell's equation for $\nabla \times \mathbf{B}$, use the same vector identity on the LHS and Maxwell's equation for $\nabla \times \mathbf{E}$. The result is the *same wave equation* for $\mathbf{B}$. Just going through these steps we obtain

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{B}) &= \frac{1}{c^2}\nabla \times \left(\frac{\partial \mathbf{E}}{\partial t}\right) \\ \Rightarrow -\nabla^2 \mathbf{B} &= \frac{1}{c^2}\frac{\partial(\nabla \times \mathbf{E})}{\partial t} \\ \Rightarrow -\nabla^2 \mathbf{B} &= -\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} \end{aligned} \tag{6.7}$$

So $\mathbf{B}$ does indeed satisfy

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B} = 0 \tag{6.8}$$

Maxwell's work was completed in 1864 and published in 1865; the idea that light could be an electromagnetic wave had been made as early as 1846 by Faraday under remarkable circumstances—cf. the remarks made in the preface to these lecture notes.

An important fact about one dimensional waves travelling, say, in the x-direction with velocity c. is that if $f(x,t)$ represents any function *or vector* which is a solution to the wave equation so that

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) f(x,t) = 0 \tag{6.9}$$

then

$$f = f(x - ct) \tag{6.10}$$

is always a solution for *any* [1] f—we shall see below that $f = f(x + ct)$ is also always a solution.

Example *Standing waves*

A standing wave is a superposition of two waves travelling in opposite directions. For example, in one dimension, a wave travelling with velocity c along the x-axis towards the positive direction has x, t dependence of the form

$$f(x - ct) \tag{6.11}$$

[1] For example the reader can try $f = \cos(x - ct)$ or $f = \exp(x - ct)$ and verify by explicit differentiation that these two functions satisfy the wave equation and so on.

Hence the superposition

$$f(x-ct)+g(x+ct) \tag{6.12}$$

represents two waves travelling in opposite directions and is therefore a *standing wave* (sometimes it is insisted that the functions g and f are the same for a standing wave). One should also add that it is immediate that both of the above functions are solutions to the wave equation. In other words we have

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) f(x-ct) &= 0 \\ \left(\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)(f(x-ct)+g(x+ct)) &= 0 \end{aligned} \tag{6.13}$$

as can quickly be verified by direct differentiation.

A standing wave can often have a misleading appearance as it may be written as *a product* of two functions instead of as *a sum*. If this is so then it can be rewritten as a sum and, to elucidate this point, consider the following example. Suppose

$$f(x,t)=\cos(x)\cos(ct) \tag{6.14}$$

It is trivial to check by differentiation that f satisfies the wave equation but to see that f is expressible as a sum we simply observe that the trigonometric formula

$$\cos(A)\cos(B)=\frac{1}{2}\left(\cos(A+B)+\cos(A-B)\right) \tag{6.15}$$

from which we deduce at once that

$$f(x,t)=\frac{1}{2}\cos(x-ct)+\frac{1}{2}\cos(x+ct) \tag{6.16}$$

which is precisely what we wanted.

§ 2. Transversality of E and B and the property $\mathbf{E}\cdot\mathbf{B}=0$

We now want to show that both **E** and **B** are what are called *transverse* waves. A transverse wave is one where the oscillations are perpendicular to the direction of travel so, in the current setting, we want to show that when **E** and **B** travel as waves the vectors **E** and **B** are perpendicular to the direction of travel. We shall carry out our discussion for 1-dimensional waves travelling in the x-direction—this means that we assume that there is no dependence of the wave on y and z, just a dependence on x and t.

Since $\nabla\cdot\mathbf{E}=0$ we have

$$\begin{aligned} &\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=0 \\ &\qquad\Rightarrow \frac{\partial E_x}{\partial x}=0, \quad \text{since } \mathbf{E} \text{ is independent of } y \text{ and } z \\ &\qquad\Rightarrow E_x = C, \quad \text{with } C \text{ a constant} \end{aligned} \tag{6.17}$$

However, on physical grounds, we require

$$C = 0 \tag{6.18}$$

because we wish all waves to die away at infinity and a constant field does not do this. So we have

$$\begin{aligned} E_x &= 0 \\ \Rightarrow \mathbf{E} &= E_y\,\mathbf{j} + E_z\,\mathbf{k} \\ \Rightarrow \mathbf{E}\cdot\mathbf{i} &= 0 \\ \Rightarrow \mathbf{E} & \quad \text{is transverse} \end{aligned} \tag{6.19}$$

But now we *rotate* our coordinate system about the x-axis so that the y-axis coincides with the direction of $\mathbf{E}$, this makes $E_z = 0$ so that we have, finally

$$\mathbf{E} = E_y\,\mathbf{j} \tag{6.20}$$

Next we turn our attention to $\mathbf{B}$. We also have $\nabla\cdot\mathbf{B} = 0$ so, as we just did for $\mathbf{E}$, we can deduce that

$$B_x = C, \quad C \text{ a constant} \tag{6.21}$$

and C must be zero for the same reason as we gave for $\mathbf{E}$. Hence

$$\begin{aligned} \mathbf{B} &= B_y\,\mathbf{j} + B_z\,\mathbf{k} \\ \Rightarrow \mathbf{B}\cdot\mathbf{i} &= 0, \quad \text{so } \mathbf{B} \text{ is transverse} \end{aligned} \tag{6.22}$$

Finally we want to prove that $\mathbf{E}$ is perpendicular to $\mathbf{B}$. To do this consider the Maxwell equation

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{6.23}$$

Using $\mathbf{E} = E_y\,\mathbf{j}$ and $\mathbf{B} = B_y\,\mathbf{j} + B_z\,\mathbf{k}$ we obtain

$$\begin{aligned} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & E_y & 0 \end{vmatrix} &= -\frac{\partial\mathbf{B}}{\partial t} \\ \Rightarrow \frac{\partial E_y}{\partial x}\,\mathbf{k} &= -\frac{\partial B_y}{\partial t}\mathbf{j} - \frac{\partial B_z}{\partial t}\mathbf{k} \end{aligned} \tag{6.24}$$

and so we must have

$$\frac{\partial B_y}{\partial t} = 0 \tag{6.25}$$

and this allows only a constant B_y which we reject as before. Hence we have found that

$$\mathbf{B} = B_z\,\mathbf{k} \tag{6.26}$$

so that $\mathbf{E}$ and $\mathbf{B}$ are indeed perpendicular.

Summarising the various properties of wave solutions to Maxwell's equations we have found that both **E** and **B** are transverse and that $\mathbf{E} \cdot \mathbf{B} = 0$. This then means that if a one dimensional wave travels in the x-direction, with **E** along the y-axis, then we have

$$\mathbf{E} = E_y\,\mathbf{j}, \quad \mathbf{B} = B_z\,\mathbf{k} \tag{6.27}$$

giving automatically

$$\mathbf{E} \cdot \mathbf{B} = 0 \tag{6.28}$$

Example *A simple electric wave*

Suppose $\mathcal{E}$ is a constant and

$$\mathbf{E} = \mathcal{E}\cos(kx - \omega t)\,\mathbf{j}, \qquad \text{with } k = \frac{2\pi}{\lambda} \qquad \text{and } \omega = 2\pi\nu \tag{6.29}$$

then **E** is a wave provided

$$\lambda\nu = c \tag{6.30}$$

The quantities λ and ν are called the *wavelength* and *frequency* of the wave respectively. To verify this we simply substitute **E** into the wave equation 6.9 and obtain

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\cos(kx - \omega t)\,\mathbf{j} = (-k^2 + \frac{\omega^2}{c^2})\cos(kx - \omega t)\,\mathbf{j} \tag{6.31}$$

Hence, for **E** to be a solution,. we must have

$$\begin{aligned} k^2 &= \frac{\omega^2}{c^2} \\ \Rightarrow \frac{(2\pi)^2}{\lambda^2} &= \frac{(2\pi\nu^2)}{c^2} \\ \Rightarrow \lambda^2\nu^2 &= c^2 \\ \text{or} \qquad \lambda\nu &= c \end{aligned} \tag{6.32}$$

as claimed.

We can also write **E** in the form $f(x - ct)$ for note that we have

$$\begin{aligned} \mathbf{E} &= \mathcal{E}\cos\left(\frac{2\pi}{\lambda}x - 2\pi\nu t\right)\,\mathbf{j} \\ &= \mathcal{E}\cos\left\{\frac{2\pi}{\lambda}(x - ct)\right\}\,\mathbf{j} \\ &= f(x - ct) \\ \text{with} \qquad f(x - ct) &= \mathcal{E}\cos\left\{\frac{2\pi}{\lambda}(x - ct)\right\}\,\mathbf{j} \end{aligned} \tag{6.33}$$

§ 3. The flow of energy for electromagnetic waves

We wish, in this section, to pin down the energy properties of electromagnetic waves and discover how they can flow through a vacuum as radiation. We shall find that Maxwell's equations do, more or less, all this for us.

We begin with Maxwell's equation

$$\nabla \times \mathbf{B} = \frac{1}{\epsilon_0 c^2}\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{6.34}$$

which we rewrite as

$$\mathbf{J} = \epsilon_0 c^2 \nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{6.35}$$

Dotting with $\mathbf{E}$ gives

$$\begin{aligned} \mathbf{E}\cdot\mathbf{J} &= \epsilon_0 c^2 \mathbf{E}\cdot\nabla\times\mathbf{B} - \epsilon_0 \mathbf{E}\cdot\frac{\partial \mathbf{E}}{\partial t} \\ &= \epsilon_0 c^2 \mathbf{E}\cdot\nabla\times\mathbf{B} - -\frac{\epsilon_0}{2}\frac{\partial}{\partial t}(\mathbf{E}^2) \end{aligned} \tag{6.36}$$

But using the vector identity

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B} \tag{6.37}$$

on $\mathbf{E}$ and $\mathbf{B}$ in the form

$$\mathbf{E}\cdot\nabla\times\mathbf{B} = -\nabla\cdot(\mathbf{E}\times\mathbf{B}) + \mathbf{B}\cdot\nabla\times\mathbf{E} \tag{6.38}$$

we get

$$\mathbf{E}\cdot\mathbf{J} = -\epsilon_0 c^2 \nabla\cdot(\mathbf{E}\times\mathbf{B}) + \epsilon_0 c^2 \mathbf{B}\cdot\nabla\times\mathbf{E} - \frac{\epsilon_0}{2}\frac{\partial}{\partial t}(\mathbf{E}^2) \tag{6.39}$$

However

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{6.40}$$

and making this substitution on the RHS of the expression for $\mathbf{E}\cdot\mathbf{J}$ we find that

$$\begin{aligned} \mathbf{E}\cdot\mathbf{J} &= -\epsilon_0 c^2 \nabla\cdot(\mathbf{E}\times\mathbf{B}) - \epsilon_0 c^2 \mathbf{B}\cdot\frac{\partial}{\partial t}(\mathbf{B}) - \frac{\epsilon_0}{2}\frac{\partial}{\partial t}(\mathbf{E}^2) \\ &= -\epsilon_0 c^2 \nabla\cdot(\mathbf{E}\times\mathbf{B}) - \frac{\epsilon_0 c^2}{2}\cdot\frac{\partial}{\partial t}(\mathbf{B}^2) - \frac{\epsilon_0}{2}\frac{\partial}{\partial t}(\mathbf{E}^2) \\ &= -\epsilon_0 c^2 \nabla\cdot(\mathbf{E}\times\mathbf{B}) - \frac{\epsilon_0}{2}\frac{\partial}{\partial t}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) \end{aligned} \tag{6.41}$$

Hence if we define the vector $\mathbf{S}$ and the scalar u by writing

$$\mathbf{S} = \epsilon_0 c^2(\mathbf{E}\times\mathbf{B}), \quad u = \frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) \tag{6.42}$$

then $\mathbf{E}\cdot\mathbf{J}$ satisfies the equation

$$\mathbf{E}\cdot\mathbf{J} = -\nabla\cdot\mathbf{S} - \frac{\partial u}{\partial t} \tag{6.43}$$

To understand the physical meaning of this result we integrate both sides over an arbitrary volume V obtaining

$$\int_V \mathbf{E}\cdot\mathbf{J}\,dV = -\int_V \nabla\cdot\mathbf{S}\,dV - \frac{\partial}{\partial t}\int_V u\,dV \tag{6.44}$$

Now u has the dimension of energy per unit volume—i.e. it has the dimensions of an *energy density*—so we write this equation as

$$-\frac{\partial}{\partial t}\int_V u\,dV = \int_S \mathbf{S}\cdot\mathbf{ds} + \int_V \mathbf{E}\cdot\mathbf{J}\,dV \tag{6.45}$$

where we also used Gauss's divergence theorem on the $\mathbf{S}$ integral. But the LHS of 6.45 is identifiable as the decrease per unit time of the *total* energy in V and $\int_V \mathbf{E}\cdot\mathbf{J}\,dV$ is the work done on any charges inside V. Hence, conservation of energy means that the term

$$\int_S \mathbf{S}\cdot\mathbf{ds} \tag{6.46}$$

must represent the flow inwards or outwards of any energy entering or leaving V across its surface.

In fact u is the energy density of the radiation of the electromagnetic field and $\mathbf{S}$ is called *Poynting's vector*. The physical interpretation of Poynting's vector[2] is that the flow of energy per unit area, per unit time, in a direction $\mathbf{n}$, where $\mathbf{n}$ is a unit vector is given by

$$\mathbf{S}\cdot\mathbf{n} \tag{6.47}$$

Note that $\mathbf{S}$ points in the direction of travel of the wave so that the flow of energy is in the right direction.

The reader should also be able to see by now that an electromagnetic wave is *a pair* of waves, one electric and one magnetic, simultaneously travelling with the velocity of light; further both waves are transverse and one is perpendicular to the other. What we have really done here—and this is really very important and a great triumph of Maxwell's equations—is to elucidate fully the electromagnetic nature of *any* light wave.

In sum then the energy properties of the waves making up $\mathbf{E}$ and $\mathbf{B}$ are

$$\begin{aligned} u = \frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right), &\quad \text{(the energy density)} \\ \mathbf{S}\cdot\mathbf{n}, &\quad \text{(the flow of energy}/m^2/s \text{ in direction } \mathbf{n}\ (\mathbf{n}^2 = 1)) \\ \text{where } \mathbf{S} = \epsilon_0 c^2(\mathbf{E}\times\mathbf{B}) & \end{aligned} \tag{6.48}$$

[2] Poynting's vector is also written in many experimental physics texts as

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}$$

where $\mathbf{H} = \epsilon_0 \mathbf{c}^2\mathbf{B}$ but we cannot go into the reason for that here.

Example *The expressions u and* **S** *for a simple wave*

We shall calculate here the quantities u and $\mathbf{S}$ for a simple wave. Let $\mathcal{E}$ and $\mathcal{B}$ be constants and k and ω have their usual meaning. Then consider the electromagnetic wave

$$\begin{aligned}\mathbf{E} &= \mathcal{E}\cos(kx-\omega t)\,\mathbf{j}\\ \mathbf{B} &= \mathcal{B}\cos(kx-\omega t)\,\mathbf{k}\end{aligned} \tag{6.49}$$

then for u and $\mathbf{S}$ we find the expressions

$$\begin{aligned}u &= \frac{\epsilon_0}{2}\left(\mathcal{E}^2+c^2\mathcal{B}^2\right)\cos^2(kx-\omega t)\\ \mathbf{S} &= \epsilon_0 c^2\mathcal{E}\mathcal{B}\cos^2(kx-\omega t)\,\mathbf{i}\end{aligned} \tag{6.50}$$

As an exercise the reader should use Maxwell's equations to show that the constants $\mathcal{E}$ and $\mathcal{B}$ are related and that in fact

$$\mathcal{B} = \frac{\mathcal{E}}{c} \tag{6.51}$$

thus one of $\mathcal{E}$ or $\mathcal{B}$ can be eliminated in the expressions above for u and $\mathbf{S}$.

Example *The average energy per cycle*

Since a wave, in general, repeats itself every cycle, i.e. every $1/\nu$ seconds, it is more useful when quoting $\mathbf{S}\cdot\mathbf{n}$ to give its *average* over one compete cycle. Let us calculate this average for the wave of the previous example.

First we denote the average of $\mathbf{S}\cdot\mathbf{n}$ over one cycle by $<\mathbf{S}\cdot\mathbf{n}>$ where $<\mathbf{S}\cdot\mathbf{n}>$ is defined by

$$<\mathbf{S}\cdot\mathbf{n}>= \frac{\int_0^T \mathbf{S}\cdot\mathbf{n}\,dt}{T}, \qquad \text{where } T=\frac{1}{\nu}=\frac{2\pi}{\omega}, \quad \text{since } \omega=2\pi\nu \tag{6.52}$$

Now let us choose, for simplicity, $\mathbf{n}=\mathbf{i}$ so that the expressions from the previous example give us

$$\begin{aligned}\mathbf{S}\cdot\mathbf{i} &= \epsilon_0 c^2\mathcal{E}\mathcal{B}\cos^2(kx-\omega t)\\ \Rightarrow <\mathbf{S}\cdot\mathbf{i}> &= \epsilon_0 c^2\mathcal{E}\mathcal{B}\frac{1}{T}\int_0^T \cos^2(kx-\omega t)\,dt, \quad T=\frac{2\pi}{\omega}\end{aligned} \tag{6.53}$$

To evaluate the integral[3] note that T is just a complete period and so

$$\begin{aligned}\frac{1}{T}\int_0^T \cos^2(kx-\omega t)\,dt &= \frac{1}{2\pi}\int_0^{2\pi}\cos^2(\theta)\,d\theta\\ &= \frac{1}{2}\end{aligned} \tag{6.56}$$

[3] If the reader wants the details spelled out we give them in this footnote; however we do not require them for this course. Setting $\theta = kx-\omega t$ we find that

$$\frac{1}{T}\int_0^T \cos^2(kx-\omega t)\,dt = -\frac{1}{\omega T}\int_{kx}^{(kx-\omega T)}\cos^2(\theta)\,d\theta$$

Now we can substitute for the integral in the expression for $< \mathbf{S} \cdot \mathbf{i} >$ and obtain the final result which is

$$\begin{aligned} < \mathbf{S} \cdot \mathbf{i} > &= \frac{\epsilon_0 c^2 \mathcal{E} \mathcal{B}}{2} \\ &= \frac{1}{2} \epsilon_0 c \mathcal{E}^2, \quad \text{if we use the result } \mathcal{B} = \frac{\mathcal{E}}{c} \end{aligned} \tag{6.57}$$

Example *Poynting's vector* **S** *in practice: power dissipation in a wire*

Let us take a conducting wire of resistance R and apply a potential difference V to it producing a current I. Then we immediately have an electric field $\mathbf{E}$ inside the wire related to V by

$$V = |\mathbf{E}|L \tag{6.58}$$

and a magnetic field $\mathbf{B}$ which at a distance r from the wire we know is given by the expression

$$\begin{aligned} \mathbf{B} &= \frac{\mu_0 I}{2\pi r} \mathbf{e} \\ &= \frac{I}{2\pi \epsilon_0 c^2 r} \mathbf{e}, \quad \text{using } \mu_0 = \frac{1}{\epsilon_0 c^2} \end{aligned} \tag{6.59}$$

where $\mathbf{e}$ is a unit tangent vector to the line of force of radius r. Now take a cylinder of length L and radius r that encloses the wire. Since $\mathbf{S} = \epsilon_0 c^2 \mathbf{E} \times \mathbf{B}$ is radial then the energy escaping per unit time from the wire is expressible as an integral over the curved surface S of this cylinder. If we denote this quantity by P, since it is actually the power dissipated by the wire, then we have

$$P = \int_S \mathbf{S} \cdot \hat{\mathbf{r}} = \epsilon_0 c^2 |\mathbf{E} \times \mathbf{B}| 2\pi r L, \quad \text{since } \mathbf{E} \times \mathbf{B} \text{ is constant on } S \tag{6.60}$$

But, if $\mathbf{n}$ denotes a unit vector along the wire, then

$$\mathbf{E} = |\mathbf{E}|\mathbf{n} \tag{6.61}$$

But is easy to verify that

$$\begin{aligned} &\int \cos^2(\theta)\, d\theta = \{\sin(\theta)\cos(\theta) + \theta\} \\ \Rightarrow -\frac{1}{\omega T} &\int_{kx}^{(kx-\omega T)} \cos^2(\theta)\, d\theta = -\frac{1}{2\omega T} [\sin(\theta)\cos(\theta) + \theta]_{kx}^{(kx-\omega T)} \end{aligned} \tag{6.54}$$

and when we compute the RHS of 6.54 we obtain

$$-\frac{1}{2\omega T} \{\sin(kx - \omega T)\cos(kx - \omega T) - \sin(kx)\cos(kx) - \omega T\} = \frac{1}{2} \tag{6.55}$$

where the cos and sin terms completely cancel with one another because $\omega T = 2\pi\nu\nu^{-1} = 2\pi$ and both are periodic with period 2π.

so that, if we use the expressions for **E** and **B** and the fact that **E** is perpendicular to **B** then we can compute that

$$\begin{aligned} P &= \epsilon_0 c^2 |\mathbf{E}| \frac{I}{2\pi\epsilon_0 c^2 r} 2\pi r L \\ &= |\mathbf{E}| I L \end{aligned} \tag{6.62}$$

But $V = |\mathbf{E}|L$ so we obtain the result

$$P = VI = I^2 R, \quad \text{using } V = RI \tag{6.63}$$

Thus we see that the Poynting vector has recovered for us the expression we already calculated by a more pedestrian method, cf. 3.32.

Example *The energy density u in practice: energy storage by a capacitor*

First a couple of facts about about capacitors—objects which were not dealt with earlier in these lectures.

A capacitor is a device for storing charge and it usually consists of two plates separated by an insulator. When a voltage difference V is applied across the plates a charge Q accumulates and the energy U required to do this is then stored in the capacitor. The fundamental relations between V, Q and U are

(i)

$$Q = CV \tag{6.64}$$

where C is a constant called the *capacitance* of the capacitor and

(ii)

$$U = \frac{1}{2} C V^2 \tag{6.65}$$

Now we shall use the energy density

$$u = \frac{\epsilon_0}{2} \left(\mathbf{E}^2 + c^2 \mathbf{B}^2 \right) \tag{6.66}$$

to calculate the energy stored in a standard parallel plate capacitor—cf. Figure 9—whose plates have area A and are separated by a distance d

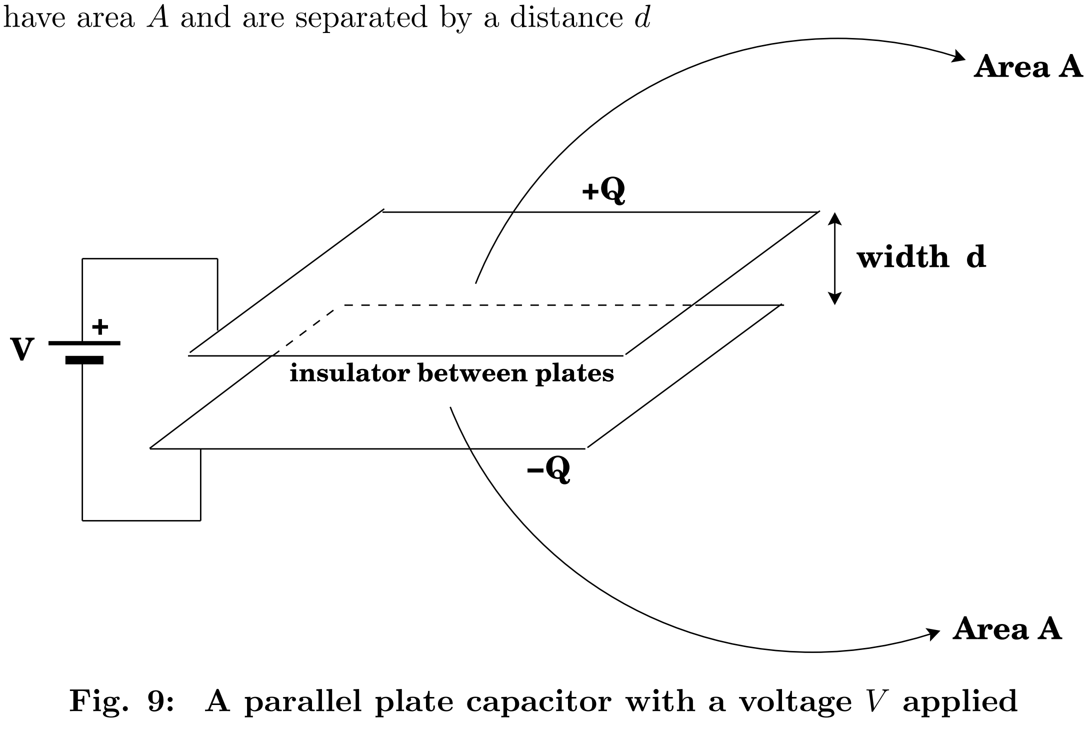

Fig. 9: A parallel plate capacitor with a voltage V applied

First of all there is no magnetic field inside the capacitor so we have

$$u = \frac{\epsilon_0}{2}\mathbf{E}^2 \tag{6.67}$$

The energy U stored inside is got by integrating u over the internal volume of the capacitor; thus

$$U = \int_V u\, dV = \frac{\epsilon_0}{2}\mathbf{E}^2 Ad, \quad \text{since } |\mathbf{E}| \text{ is constant} \tag{6.68}$$

Let us now accept that the voltage V across the capacitor is related to $\mathbf{E}$ by the equation

$$|\mathbf{E}| = \frac{V}{d} \tag{6.69}$$

Hence

$$U = \frac{\epsilon_0}{2}\frac{V^2}{d^2}Ad = \frac{1}{2}\frac{\epsilon_0 A}{d}V^2 \tag{6.70}$$

But for a parallel plate capacitor it is known that (again, we just accept this fact here)

$$C = \frac{A\epsilon_0}{d} \tag{6.71}$$

so we immediately have the result that

$$U = \frac{1}{2}CV^2 \tag{6.72}$$

and we are in agreement with our previous result, cf. 6.65.

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