Maynooth University
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MATHEMATICAL PHYSICS

SEMESTER 2, REPEAT

2018–2019

MP204
Electricity and Magnetism

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Time allowed: 2 hours

Answer **ALL FOUR** questions

1. (a) Consider the static electric field $\mathbf{E} = \lambda x \hat{\imath} - 3\lambda L \hat{\jmath}$.

   Here λ and L are positive constants.

   Find the electric potential difference V_{QP} between the point Q at $(0, 4L, 0)$ and the point P at $(0, L, 0)$.

   Find the electric potential difference V_{RP} between the point R at $(3L, L, 0)$ and the point P at $(0, L, 0)$.

   [10 marks]

   (b) A thin rod of length L lies along the x axis with one end at $\left(-\frac{L}{2}, 0, 0\right)$ and the other end at $\left(\frac{L}{2}, 0, 0\right)$. The rod carries a uniformly distributed postive charge Q. Calculate the electric field at the point $(x_0, 0, 0)$ on the x axis, with $x_0 > L/2$.

   [15 marks]

2. (a) The Biot-Savart law can be written as

$$d\mathbf{B} = \left(\frac{\mu_0 I}{4\pi}\right) \frac{d\mathbf{l}' \times \left(\widehat{\mathbf{r}-\mathbf{r}'}\right)}{|\mathbf{r}-\mathbf{r}'|^2}$$

Explain what this equation expresses. Each quantity in the equation should be explained clearly.

Include an appropriate figure or figures, showing relevant vectors and/or distances. Indicate the direction of $d\mathbf{B}$ in your sketch.

[11 marks]

(b) A ring of wire with radius R is centered at the origin and lies on the x-y plane, so that its axis coincides with the z-axis. Current I_1 flows around this circular wire loop.

Using the Biot-Savart law, calculate the magnetic field at the center of the loop, i.e., at the origin.

[14 marks]

3. (a) The vector potential in some region is given by

$$\mathbf{A} = \left(-\lambda\frac{z}{2}\right)\hat{\jmath} + \left(\lambda\frac{y}{2}\right)\hat{k}$$

Find the magnetic field **B**.

Consider adding ∇f to the vector potential, where f is any scalar function. Explain how the magnetic field changes due to this transformation.

[8 marks]

(b) A long solenoid has n turns per unit length and radius R. It carries time-varying current $I(t) = I_0 \sin(\omega t)$. Use Faraday's law in integral form to calculate the induced electric field as a function of the distance r from the axis of the solenoid, at a point outside the solenoid ($r > R$).

Reminder: A current I through a solenoid creates a magnetic field $\mu_0 n I$ inside the solenoid, where n is the number of turns per unit length.

[17 marks]

4. (a) The magnetic field in a region changes with time, $\mathbf{B} = B_0 e^{-2t/t_0}\hat{\imath}$. Here B_0 and t_0 are positive constants. We consider two square loops, which we call Γ_1 and Γ_2.

   Γ_1 lies in the y-z plane, and has sides of length L_1. Find the electromotive force (EMF) induced in this loop.

   The other square loop, Γ_2, lies in the x-y plane, and has sides of length L_2. Find the EMF induced in Γ_2.

[10 marks]

(b) Consider an electromagnetic wave travelling through empty space described by the electric and magnetic fields

$$\mathbf{E} = 4V_0 L \cos\left(\frac{1}{L}(y - ct)\right)\hat{k}\,, \qquad \mathbf{B} = \mathbf{G}\cos\left(\frac{1}{L}(y - ct)\right)\,,$$

where V_0 and L are positive constants and $\mathbf{G}$ is a constant vector.

Using Maxwell's equations in free space, find the constant vector $\mathbf{G}$.

In which direction is the electromagnetic wave propagating?

[10 marks]

(c) A magnetic field of $\mathbf{B} = a\sin(by)e^{bx}\hat{k}$ is produced by a steady electric current. What is the density of that current?

[5 marks]

Posssibly useful Equations

- Electrostatics: $\mathbf{E} = -\nabla V \quad ; \quad V_{PQ} = -\int_Q^P \mathbf{E} \cdot d\mathbf{l}$

  Electric potential at $\mathbf{r}$ due to a point charge q_1 at $\mathbf{r_1}$: $V = \dfrac{q_1}{4\pi\epsilon_0} \dfrac{1}{|\mathbf{r} - \mathbf{r_1}|}$

  Gauss' law: $\oint_\Sigma \mathbf{E} \cdot d\mathbf{S} = \dfrac{Q_{\text{enclosed}}}{\epsilon_0}$

- Magnetostatics: Ampere's law: $\int_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enclosed}}$

  Biot-Savart law: $d\mathbf{B} = \left(\dfrac{\mu_0 I}{4\pi}\right) \dfrac{d\mathbf{l}' \times \left(\widehat{\mathbf{r} - \mathbf{r}'}\right)}{|\mathbf{r} - \mathbf{r}'|^2}$

- Force on a charge: $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$

  Magnetic force on a current element: $d\mathbf{F} = I d\mathbf{l} \times \mathbf{B}$

- Fields from potentials: $\mathbf{E} = -\nabla V - \dfrac{\partial \mathbf{A}}{\partial t} \quad , \quad \mathbf{B} = \nabla \times \mathbf{A}$

- The continuity equation: $\nabla \cdot \mathbf{J} + \dfrac{\partial \rho}{\partial t} = 0$

- Faraday's law: $\mathcal{E} = -\dfrac{d\Phi_B}{dt}$ or $\oint_C \mathbf{E} \cdot d\mathbf{l} = -\dfrac{d}{dt}\Phi_B = -\dfrac{d}{dt}\int_\Sigma \mathbf{B} \cdot d\mathbf{S}$

- Maxwell's Equations:

  ① $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}$ ② $\nabla \cdot \mathbf{B} = 0$

  ③ $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$

  ④ $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} = \mu_0 (\mathbf{J} + \mathbf{J_D})$

- Poynting vector: $\mathbf{S} = \dfrac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$ Speed of light: $c = 1/\sqrt{\mu_0\epsilon_0}$

  Energy density of electromagnetic fields: $u = \dfrac{1}{2}\epsilon_0 E^2 + \dfrac{1}{2\mu_0} B^2$