MP204
Electricity and Magnetism

2017–2018, practice exam 3

Time allowed: 2 hours

Answer **ALL** questions

This is a **SAMPLE** exam, roughly reflecting the general structure of the MP204 exams for 2017 – 2018.

Remember that **ALL** questions are to be answered.

If any figures are relevant, please include them with your solution! Sketching appropriate pictures often helps you arrive at the correct solution.

1. (a) An infinite charged plane carries surface charge density σ. Use Gauss' dielectric flux theorem to calculate the electric field at distance z from the plane. Explain very clearly the shape of the closed surface on which you apply the theorem. Plot the electric field magnitude as a function of z.

[12 marks]

(b) The electrostatic field in some region is given by

$$\mathbf{E} = (2x\ W/L^2)\hat{\imath} + (5\ W/L)\hat{k}$$

where W and L are positive constants.

Find the potential difference between the points $(L, 3L, 0)$ and $(4L, L, 0)$.

Find the charge density in the region.

What is the electric flux through a surface lying parallel to the x-y plane and having area $3L^2$?

[18 marks]

2. (a) A long straight wire carries steady current I. The wire cross-section is circular and has radius R. The current density is uniform inside the wire. Using Ampere's law, calculate the magnetic field at distance r from the axis of the wire. Consider separately the cases $r < R$ (inside the wire) and $r > R$ (outside the wire). Plot the magnitude of the magnetic field as a function of r.

[14 marks]

(b) An infinite wire carrying current I runs along the y axis; the current flows from $y = -\infty$ to $y = +\infty$ through the origin. A rectangular loop lies in the xy plane, with the four corners having coordinates $(3L, 0)$, $(5L, 0)$, $(5L, L)$, and $(3L, L)$. Find the magnetic flux through the rectangular loop.

Reminder: An infinitely long straight wire produces a magnetic field of strength $\mu_0 I/(2\pi d)$ at a point at distance d from the wire.

[16 marks]

(c) The magnetic field in some region is uniform: $\mathbf{B} = B_0\hat{k}$. At time $t = 0$, a charged particle (mass m, charge q) is at the origin and has velocity $\mathbf{v} = \alpha\hat{\jmath}$.

Describe the motion (trajectory) of the particle. Sketch the trajectory, indicating clearly the coordinates of at least two points on the trajectory.

[10 marks]

3. (a) An electromagnetic system is described by the scalar and vector potentials

$$V = 3E_0 L e^{-x^2/L^2} e^{-2\omega t} , \qquad \mathbf{A} = \frac{E_0}{\omega} \sin\left(\frac{\omega}{c} y - \omega t\right) \hat{k}$$

where L, E_0 and ω are positive constants.

Calculate the electric and magnetic fields.

Calculate the charge density, the current density, and the displacement current density.

[20 marks]

(b) Using Maxwell's equations in vacuum, derive a continuity equation for the energy carried by electromagnetic fields, in terms of the Poynting vector. The vector identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

might be helpful.

[10 marks]

Posssibly useful Equations

- Electrostatics: $\mathbf{E} = -\nabla V \quad ; \quad V_{PQ} = -\int_Q^P \mathbf{E} \cdot d\mathbf{l}$

  Electric potential at $\mathbf{r}$ due to a point charge q_1 at $\mathbf{r_1}$: $V = \dfrac{q_1}{4\pi\epsilon_0} \dfrac{1}{|\mathbf{r} - \mathbf{r_1}|}$

  Gauss' law: $\oint_\Sigma \mathbf{E} \cdot d\mathbf{S} = \dfrac{Q_{\text{enclosed}}}{\epsilon_0}$

- Magnetostatics: Ampere's law: $\int_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enclosed}}$

  Biot-Savart law: $d\mathbf{B} = \left(\dfrac{\mu_0 I}{4\pi}\right) \dfrac{d\mathbf{l}' \times \left(\widehat{\mathbf{r} - \mathbf{r}'}\right)}{|\mathbf{r} - \mathbf{r}'|^2}$

- Force on a charge: $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$

  Magnitic force on a current element: $d\mathbf{F} = I d\mathbf{l} \times \mathbf{B}$

- Fields from potentials: $\mathbf{E} = -\nabla V - \dfrac{\partial \mathbf{A}}{\partial t} \quad , \quad \mathbf{B} = \nabla \times \mathbf{A}$

- The continuity equation: $\nabla \cdot \mathbf{J} + \dfrac{\partial \rho}{\partial t} = 0$

- Maxwell's Equations:

$$① \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad ② \quad \nabla \cdot \mathbf{B} = 0$$

$$③ \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$④ \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 (\mathbf{J} + \mathbf{J_D})$$

- Poynting vector: $\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$ Speed of light: $c = 1/\sqrt{\mu_0 \epsilon_0}$

  Energy density of electromagnetic fields: $u = \dfrac{1}{2}\epsilon_0 E^2 + \dfrac{1}{2\mu_0} B^2$