MP204
Electricity and Magnetism

2017–2018, practice exam 2

Time allowed: 2 hours

Answer **ALL** questions

This is a **SAMPLE** exam, roughly reflecting the general structure of the MP204 exams for 2017 – 2018.

Remember that **ALL** questions are to be answered.

If any figures are relevant, please include them with your solution! Sketching appropriate pictures often helps you arrive at the correct solution.

1. (a) A thin rod of length $2L$ lies along the x axis with one end at $(-L, 0, 0)$ and the other end at $(L, 0, 0)$. The rod carries a uniformly distributed postive charge Q.

   Calculate the electric field generated by the charged rod at the point $(0, y_0, 0)$ on the y axis.

   You might need the integral

$$\int \frac{du}{(u^2 + a^2)^{3/2}} = \frac{u}{a^2\sqrt{u^2 + a^2}} \, .$$

[15 marks]

   (b) Consider the static electric field $\mathbf{E} = 4\alpha x^2 \hat{\imath} + 2\alpha(y + L)^2 \hat{\jmath}$.

   Here α and L are positive constants.

   Find the electric potential difference V_{QP} between the point Q at $(2L, L, 0)$ and the point P at $(L, 0, 0)$.

[15 marks]

2. (a) Steady current I flows through an infinitely long straight wire placed along the z axis, from $z = -\infty$ to $z = +\infty$ through the origin.

   At the point $(-4L, 3L, 0)$ in the x-y plane, find the magnitude of the magnetic field created by the current. Find also the x-, y-, and z- components of the magnetic field at this point.

   Now consider a point $(-4L, 3L, 2L)$, not in the x-y plane. Write down the x-, y-, and z- components of the magnetic field at this point.

   Reminder: An infinitely long straight wire produces a magnetic field of strength $\mu_0 I/(2\pi d)$ at a point at distance d from the wire.

   [16 marks]

   (b) Ampere's theorem for the magnetic field due to a steady current is

$$\oint_C \mathbf{B} \cdot d\mathbf{l} \; = \; \mu_0 I$$

   What does C denote? What does the symbol I refer to?

   Starting from this integral relation, derive part of Maxwell's fourth equation (which concerns the curl of of the magnetic field). If you use a theorem about vector integrals on the way, name the theorem explicitly.

   [12 marks]

   (c) The current density in some region is described by

$$\mathbf{J} = \gamma \left(\frac{4y}{x^2 + y^2} \hat{\imath} - \frac{4x}{x^2 + y^2} \hat{\jmath} \right)$$

   How is the charge density ρ in this region changing? If you use an equation related to the conservation of charge, name this equation explicitly.

   [7 marks]

3. (a) An electromagnetic field is described by the scalar potential V and vector potential $\mathbf{A}$, given by

$$V = -LB_0\omega x\cos(\omega t)\,; \quad \mathbf{A} = B_0 y\sin(\omega t)\hat{\imath}$$

where B_0, L, and ω are positive constants. Find the electric and magnetic fields in this system. In which spatial positions (which part of space) does the electric field vanish?

Use Maxwell's equations to calculate the current density and the charge density in this system.

[18 marks]

(b) The magnetic field in a region changes with time, $\mathbf{B} = B_0 e^{-2t/t_0}\hat{\imath}$. Here B_0 and t_0 are positive constants.

A square loop lies in the y-z plane, and has sides of length L_1. Find the electromotive force (EMF) induced in this loop.

[8 marks]

(c) Consider the electromagnetic wave in free space

$$\mathbf{E} = E_0\cos\left(\frac{2\pi c}{\lambda}t\right)\sin\left(\frac{2\pi}{\lambda}y\right)\hat{k}$$
$$\mathbf{B} = -(E_0/c)\sin\left(\frac{2\pi c}{\lambda}t\right)\cos\left(\frac{2\pi}{\lambda}y\right)\hat{\imath}$$

where λ is a positive constant.

Is this wave traveling?

Calculate the Poynting vector and explain the direction of energy flow using your result.

[9 marks]

Posssibly useful Equations

- Electrostatics: $\mathbf{E} = -\nabla V$; $V_{PQ} = -\int_Q^P \mathbf{E}\cdot d\mathbf{l}$

  Electric potential at $\mathbf{r}$ due to a point charge q_1 at $\mathbf{r_1}$: $V = \dfrac{q_1}{4\pi\epsilon_0}\dfrac{1}{|\mathbf{r}-\mathbf{r_1}|}$

  Gauss' law: $\oint_\Sigma \mathbf{E}\cdot d\mathbf{S} = \dfrac{Q_{\text{enclosed}}}{\epsilon_0}$

- Magnetostatics: Ampere's law: $\int_C \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enclosed}}$

  Biot-Savart law: $d\mathbf{B} = \left(\dfrac{\mu_0 I}{4\pi}\right)\dfrac{d\mathbf{l}' \times \left(\widehat{\mathbf{r}-\mathbf{r}'}\right)}{|\mathbf{r}-\mathbf{r}'|^2}$

- Force on a charge: $\mathbf{F} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}$

  Magnitic force on a current element: $d\mathbf{F} = I d\mathbf{l}\times\mathbf{B}$

- Fields from potentials: $\mathbf{E} = -\nabla V - \dfrac{\partial \mathbf{A}}{\partial t}$, $\mathbf{B} = \nabla\times\mathbf{A}$

- The continuity equation: $\nabla\cdot\mathbf{J} + \dfrac{\partial\rho}{\partial t} = 0$

- Maxwell's Equations:

$$① \quad \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad ② \quad \nabla\cdot\mathbf{B} = 0$$

$$③ \quad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$④ \quad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} = \mu_0\left(\mathbf{J} + \mathbf{J_D}\right)$$

- Poynting vector: $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$ Speed of light: $c = 1/\sqrt{\mu_0\epsilon_0}$

  Energy density of electromagnetic fields: $u = \dfrac{1}{2}\epsilon_0 E^2 + \dfrac{1}{2\mu_0}B^2$