MP204
Electricity and Magnetism

2017–2018, practice exam 1

Time allowed: 2 hours

Answer **ALL** questions

This is a **SAMPLE** exam, roughly reflecting the general structure of the MP204 exams for 2017 – 2018.

Remember that **ALL** questions are to be answered.

If any figures are relevant, please include them with your solution! Sketching appropriate pictures often helps you arrive at the correct solution.

1. (a) An infinitely long charged cylinder has radius R and uniform charge density ρ. Use Gauss's dielectric flux theorem to calculate the electric field inside the cylinder, at a distance $r < R$ from its axis. If you use any assumptions based on symmetry, state them clearly.

[14 marks]

(b) A solid metallic sphere has radius R, and carries a total charge Q. The metal is an excellent conductor.

In which part of the sphere is the charge located?

Write expressions for the electric field magnitude at a distance r from the center of the sphere, both inside the sphere ($r < R$) and outside the sphere ($r > R$).

Sketch a plot of the electric field magnitude as a function of the distance r from the center of the sphere. Your plot should run from $r = 0$ to $r = 2R$.

Write expressions for the electric potential as a function of r, both inside and outside the sphere.

Sketch a plot of the electric potential as a function of r.

[16 marks]

2. (a) Current I flows through a rectangular wire loop. The loop has two sides of length L and two sides of length $2L$.

Using the Biot-Savart law, calculate the magnetic field created by the current at the center of the loop.

Possibly useful integral:

$$\int \frac{du}{(u^2+a^2)^{3/2}} = \frac{u}{a^2(u^2+a^2)^{1/2}}$$

[23 marks]

(b) An infinite wire carrying current I_1 runs along the y axis; the current flows from $y=-\infty$ to $y=+\infty$ through the origin. A square loop lies in the xy plane, with the four corners having coordinates (x_0, y_0), (x_0+L, y_0), (x_0+L, y_0+L), and (x_0, y_0+L). Current I_2 flows counterclockwise through the square loop.

Calculate the total force acting on the square loop due to the current in the long wire.

Reminders: (1) An infinitely long straight wire carrying current I_1 produces a magnetic field of strength $\mu_0 I_1/(2\pi d)$ at a point at distance d from the wire. (2) A magnetic field B perpendicular to a straight wire segment of length L carrying current I_2 exerts a force $I_2 L B$ on the wire segment.

[12 marks]

3. (a) The vector potential in some region is given by

$$\mathbf{A} = \left(-\lambda\frac{z}{2}\right)\hat{\jmath} + \left(\lambda\frac{y}{2}\right)\hat{k}$$

Find the magnetic field $\mathbf{B}$.

Consider adding ∇f to the vector potential, where f is any scalar function. Explain how the magnetic field changes due to this transformation.

Write down or derive a vector potential, different from the one above, which corresponds to the same magnetic field.

[11 marks]

(b) Due to the current through a long solenoid of radius R, the magnetic field inside the solenoid is increasing,

$$B = \beta t,$$

while the field outside the solenoid is zero. Here β is a positive constant.

Use Faraday's law in integral form to calculate the induced electric field as a function of the distance r from the axis of the solenoid, both inside and outside the solenoid.

[16 marks]

(c) An electromagnetic system is described by the fields

$$\mathbf{E} = K_0 x \sin(\omega t)\hat{\jmath} \qquad \mathbf{B} = \frac{K_0}{\omega}\cos(\omega t)\hat{k}$$

Calculate the displacement current density $\mathbf{J}_D$.

Use Maxwell's equations to calculate the current density $\mathbf{J}$.

[8 marks]

Posssibly useful Equations

- Electrostatics: $\mathbf{E} = -\nabla V$; $V_{PQ} = -\int_Q^P \mathbf{E} \cdot d\mathbf{l}$

  Electric potential at $\mathbf{r}$ due to a point charge q_1 at $\mathbf{r_1}$: $V = \dfrac{q_1}{4\pi\epsilon_0} \dfrac{1}{|\mathbf{r} - \mathbf{r_1}|}$

  Gauss' law: $\oint_\Sigma \mathbf{E} \cdot d\mathbf{S} = \dfrac{Q_{\text{enclosed}}}{\epsilon_0}$

- Magnetostatics: Ampere's law: $\int_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enclosed}}$

  Biot-Savart law: $d\mathbf{B} = \left(\dfrac{\mu_0 I}{4\pi}\right) \dfrac{d\mathbf{l}' \times \left(\widehat{\mathbf{r} - \mathbf{r}'}\right)}{|\mathbf{r} - \mathbf{r}'|^2}$

- Force on a charge: $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$

  Magnitic force on a current element: $d\mathbf{F} = I d\mathbf{l} \times \mathbf{B}$

- Fields from potentials: $\mathbf{E} = -\nabla V - \dfrac{\partial \mathbf{A}}{\partial t}$, $\mathbf{B} = \nabla \times \mathbf{A}$

- The continuity equation: $\nabla \cdot \mathbf{J} + \dfrac{\partial \rho}{\partial t} = 0$

- Maxwell's Equations:

$$① \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad ② \quad \nabla \cdot \mathbf{B} = 0$$

$$③ \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$④ \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 (\mathbf{J} + \mathbf{J_D})$$

- Poynting vector: $\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$ Speed of light: $c = 1/\sqrt{\mu_0 \epsilon_0}$

  Energy density of electromagnetic fields: $u = \dfrac{1}{2}\epsilon_0 E^2 + \dfrac{1}{2\mu_0} B^2$