MATHEMATICAL PHYSICS

SEMESTER 2

2016–2017

MP204
Electricity and Magnetism

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Time allowed: $1\frac{1}{2}$ hours

Answer **ALL** questions

1. (a) Consider the static electric field $\mathbf{E} = 5E_0\hat{\imath} + 2E_0\hat{\jmath}$.

   Find the electric potential difference V_{QP} between the point Q at $(0, 2d, 0)$ and the point P at $(0, d, 0)$.

   Find the electric potential difference V_{RP} between the point R at $(d, d, 0)$ and the point P at $(0, d, 0)$.

   The planar surface Σ_1 lies parallel to the x-y plane and has area A_1. Find the electric flux through this surface.

   [15 marks]

   (b) In some region, the electric field changes with time but the magnetic field does not:

$$\mathbf{E} = -\left(\frac{4K_0 t}{\epsilon_0}\right)\hat{k}\ ; \qquad \mathbf{B} = \mu_0 K_0\left(2y\hat{\imath} - 2x\hat{\jmath}\right)\ .$$

   (Here K_0 is a positive constant.) Find the charge density, the "displacement current" density and the current density in the region.

   [15 marks]

   (c) Steady current I flows through an infinitely long straight wire placed along the z axis, from $z = -\infty$ to $z = +\infty$ through the origin.

   At the point $(3L, 4L, 0)$ in the x-y plane, find the magnitude of the magnetic field created by the current. Find also the x-, y-, and z-components of the magnetic field at this point.

   Now consider a point $(3L, 4L, 7L)$, not in the x-y plane. Write down the x-, y-, and z- components of the magnetic field at this point.

   Reminder: An infinitely long straight wire produces a magnetic field of strength $\mu_0 I/(2\pi d)$ at a point at distance d from the wire.

   [20 marks]

2. (a) An electromagnetic field is described by the scalar potential V and vector potential $\mathbf{A}$, given by

$$V = -LB_0\omega z\cos(\omega t)\,; \quad \mathbf{A} = B_0 x\sin(\omega t)\hat{k}$$

where B_0, L, and ω are positive constants. Find the electric and magnetic fields in this system.

In which locations (which part of space) does the electric field vanish?

Show explicitly that the fields satisfy Maxwell's third equation, which concerns the curl of the electric field.

[20 marks]

(b) A circular loop lies in a plane perpendicular to a uniform magnetic field of strength B_0. The circular loop is squeezed so that its size decreases over time. Its radius is given by

$$r(t) = r_0\exp\left[-\frac{t}{T}\right] = r_0 e^{-t/T}\ ,$$

where r_0 and T are positive constants.

Find the magnitude of the electromotive force (EMF) induced in the loop.

[10 marks]

(c) A thin rod of length L lies along the x axis with one end at the origin and the other end at $(L, 0, 0)$. The rod carries a uniformly distributed postive charge Q. Calculate the electric field at the point $(x_0, 0, 0)$ on the x axis, with $x_0 > L$.

Approximate your result for $x_0 \gg L$. Explain your result physically, by comparing to the electric field due to a point charge.

[20 marks]

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HINTS AND/OR PARTIAL SOLUTIONS

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1. Question 1.

   (a) Question 1(a)

   Consider the static electric field $\mathbf{E} = 5E_0\hat{\imath} + 2E_0\hat{\jmath}$.

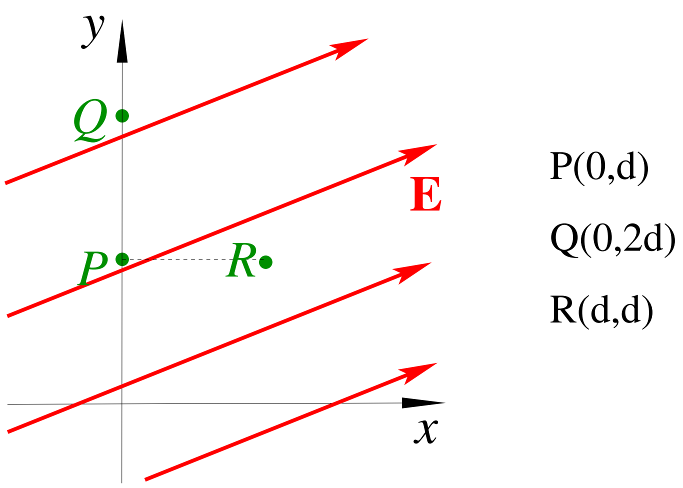
   Find the electric potential difference V_{QP} between the point Q at $(0, 2d, 0)$ and the point P at $(0, d, 0)$.

   Find the electric potential difference V_{RP} between the point R at $(d, d, 0)$ and the point P at $(0, d, 0)$.

   The planar surface Σ_1 lies parallel to the xy plane and has area A_1. Find the electric flux through this surface.

   [15 marks]

   [Sample Answer:]



   In the direction PQ (parallel to y axis), only the y-component of the electric field contributes to potential differce, hence

$$|V_{QP}| = 2E_0 \times d$$

The question is not very precise about the sign (although $V_{QP} = V_Q - V_P$ is suggested by the notation and wording); so I won't care about the sign.

More formally, if one integrates along the straight line between P and Q, the line element is $d\mathbf{l} = dy\hat{\jmath}$, so that

$$\begin{aligned} V_{QP} &= -\int_Q^P \mathbf{E}\cdot d\mathbf{l} = -\int_{2d}^{d} (5E_0\hat{\imath} + 2E_0\hat{\jmath})\cdot dy\hat{\jmath} = -2E_0\int_{2d}^{d} dy \\ &= -5E_0(-d) = 5E_0 d \end{aligned}$$

In the direction PR (parallel to x axis), only the x-component of the electric field contributes to potential differce, hence

$$|V_{RP}| = 5E_0 d$$

Electric flux: The electric field has no component in the z direction, i.e. no componennt perpendicular to the surface. Hence the flux is zero.

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(b) Question 1(b)

In some region, the electric field changes with time but the magnetic field does not:

$$\mathbf{E} = -\left(\frac{4K_0 t}{\epsilon_0}\right)\hat{k}\ ; \qquad \mathbf{B} = \mu_0 K_0\left(2y\hat{\imath} - 2x\hat{\jmath}\right)\ .$$

(Here K_0 is a positive constant.) Find the charge density, the "displacement current" density and the current density in the region.

[15 marks]

[Sample Answer:]

Using Maxwell's first equation, the charge density is

$$\rho = \epsilon_0\nabla\cdot\mathbf{E} = \epsilon_0\frac{\partial}{\partial z}\left(-\frac{4K_0 t}{\epsilon_0}\right) = 0\,.$$

The displacement current density is

$$\mathbf{J}_D \;=\; \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \;=\; -\,4K_0\hat{k}\,.$$

Maxwell's fourth equation gives for the current density

$$\mathbf{J} \;=\; \frac{1}{\mu_0}\nabla\times\mathbf{B} - \mathbf{J}_D \;=\; \frac{1}{\mu_0}\left(\mu_0 K_0(-2-2)\hat{k}\right) \;-\; \left(-4K_0\hat{k}\right) \;=\; 0.$$

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(c) Question 1(c)

Steady current I flows through an infinitely long straight wire placed along the z axis, from $z=-\infty$ to $z=+\infty$ through the origin.

At the point $(3L, 4L, 0)$ in the x-y plane, find the magnitude of the magnetic field created by the current. Find also the x-, y-, and z-components of the magnetic field at this point.

Now consider a point $(3L, 4L, 7L)$, not in the x-y plane. Write down the x-, y-, and z- components of the magnetic field at this point.

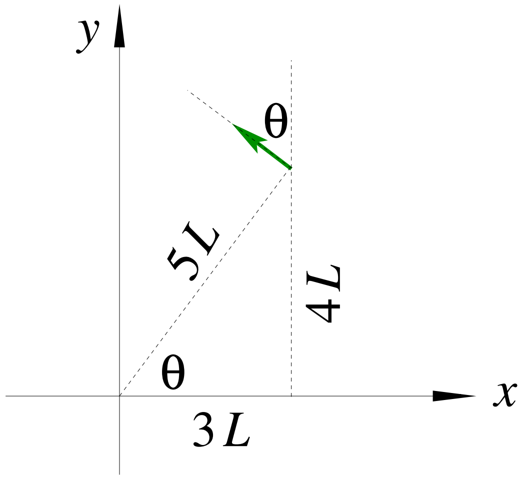
Reminder: An infinitely long straight wire produces a magnetic field of strength $\mu_0 I/(2\pi d)$ at a point at distance d from the wire.

[20 marks]

[Sample Answer:]

The distance from the wire to the point is $\sqrt{(3L)^2+(4L)^2} = 5L$, so the magnitude of the magnetic field is

$$B = \frac{\mu_0 I}{2\pi(5L)} = \frac{\mu_0 I}{10\pi L}$$



The components are

$$B_x = -B\sin\theta = -\frac{\mu_0 I}{10\pi L}\frac{4L}{5L} = -\frac{2\mu_0 I}{25\pi L}$$
$$B_y = B\cos\theta = \frac{\mu_0 I}{10\pi L}\frac{3L}{5L} = \frac{3\mu_0 I}{50\pi L}$$
$$B_z = 0$$

Now for the point $(3L, 4L, 7L)$. At a point with different z-coordinate but the same x-, y- coordinates, the magnetic field components are exactly the same by symmetry, since the wire is infinite in the z direction.

$$B_x = -\frac{2\mu_0 I}{25\pi L} \qquad B_y = \frac{3\mu_0 I}{50\pi L} \qquad B_z = 0$$

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2. Question 2.

(a) Question 2(a).
An electromagnetic field is described by the scalar potential V and vector potential $\mathbf{A}$, given by

$$V = -LB_0\omega z\cos(\omega t)\,; \quad \mathbf{A} = B_0 x\sin(\omega t)\hat{k}$$

where B_0, L, and ω are positive constants. Find the electric and magnetic fields in this system.
In which region of space does the electric field vanish?
Show explicitly that the fields satisfy Maxwell's third equation, which concerns the curl of the electric field.

[20 marks]

[Sample Answer:]
The electric and magnetic fields are

$$\begin{aligned}\mathbf{E} &= -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = LB_0\omega\cos(\omega t)\hat{k} - B_0 x\omega\cos(\omega t)\hat{k} \\ &= (L-x)B_0\omega\cos(\omega t)\hat{k}\end{aligned}$$

and

$$\begin{aligned}\mathbf{B} &= \nabla\times\mathbf{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\ &= \hat{j}\left(-\frac{\partial A_z}{\partial x}\right) = -B_0\sin(\omega t)\hat{j}\end{aligned}$$

The electric field vanishes at $x = L$. This is a plane parallel to the y-z plane.
Maxwell's third equation is $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$. Calculating the two sides for our fields,

$$\nabla\times\mathbf{E} = \hat{j}\left(-\frac{\partial E_z}{\partial x}\right) = B_0\omega\cos(\omega t)\hat{j}$$

and

$$-\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}\left[-B_0\sin(\omega t)\hat{j}\right] = B_0\omega\cos(\omega t)\hat{j}$$

The two sides are equal, hence Maxwell's third equation is satisfied.

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(b) Question 2(b).

A circular loop lies in a plane perpendicular to a uniform magnetic field of strength B_0. The circular loop is squeezed so that its size decreases over time. Its radius is given by

$$r(t) = r_0 \exp\left[-\frac{t}{T}\right] = r_0 e^{-t/T} ,$$

where r_0 and T are positive constants.

Find the magnitude of the electromotive force (EMF) induced in the loop.

[10 marks]

[Sample Answer:]

The flux of the magnetic field through the loop is

$$\Phi_B = B_0(\text{area}) = B_0\pi r^2 = B_0\pi r_0^2 e^{-2t/T}$$

The EMF according to Faraday's law is

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -B_0\pi r_0^2\,(-2/T)\,e^{-2t/T} = \frac{2B_0\pi r_0^2}{T} e^{-2t/T}$$

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(c) Question 2(c).

A thin rod of length L lies along the x axis with one end at the origin and the other end at $(L,0,0)$. The rod carries a uniformly distributed postive charge Q. Calculate the electric field at the point $(x_0,0,0)$ on the x axis, with $x_0 > L$.

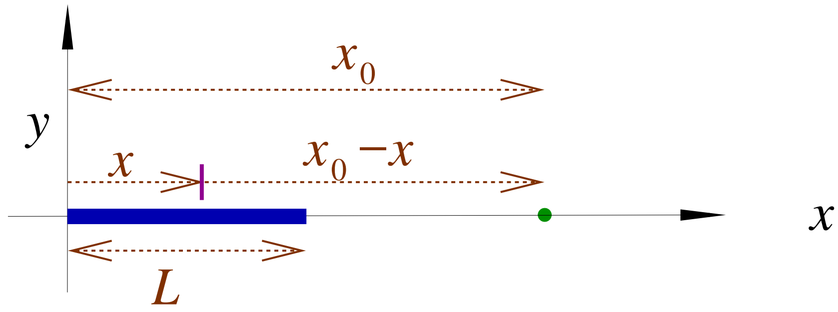
Approximate your result for $x_0 \gg L$. Explain your result physically, by comparing to the electric field due to a point charge.

[20 marks]

[Sample Answer:]

Consider an infinitesimal slice of the rod at distance x from the origin, of width dx. The amount of charge in this small piece is $(Q/L)dx$, and is it at distance $x_0 - x$ from the point $(x_0,0,0)$. The electric field due to this piece of charge is

$$d\mathbf{E} = \frac{(Q/L)dx}{4\pi\epsilon_0}\frac{\hat{i}}{(x_0-x)^2} = \frac{(Q/L)\hat{i}}{4\pi\epsilon_0}\frac{dx}{(x-x_0)^2}$$



Integrating from $x = 0$ to $x = L$ gives the total electric field created at the point $(x_0, 0, 0)$:

$$\mathbf{E} = \frac{(Q/L)\hat{i}}{4\pi\epsilon_0}\int_0^L \frac{dx}{(x-x_0)^2} = \frac{(Q/L)\hat{i}}{4\pi\epsilon_0}\left[\frac{-1}{(x-x_0)}\right]_0^L$$

$$= \frac{(Q/L)\hat{i}}{4\pi\epsilon_0}\left(\frac{1}{x_0-L} - \frac{1}{x_0}\right) = \frac{Q}{4\pi\epsilon_0}\frac{1}{x_0(x_0-L)}\,\hat{i}$$

At large distances, $x_0 \gg L$, this simplifies to

$$\mathbf{E} \approx \frac{Q}{4\pi\epsilon_0}\frac{1}{x_0^2}\,\hat{i}$$

This is the electric field due to a point charge Q at distance x_0. This makes sense because, at large enough distance, the structure of the charge-carrying stick will not be important and it will look like a small (≈point) object.

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