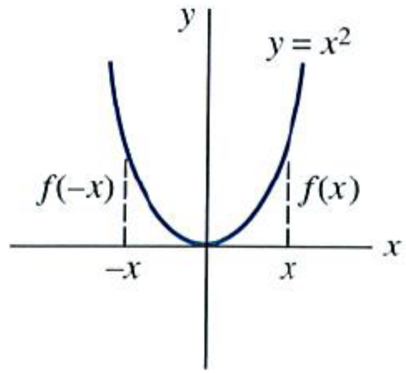
Fourier cosine and sine series

Evaluation of the coefficients in the Fourier series of a function f is considerably simpler is the function is even or odd.

A function is **even** if

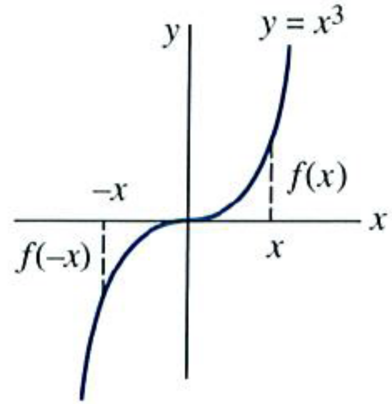
$$f(-x) = f(x)$$



Example: x^2, x^4, $\cos x$

A function is **odd** if

$$f(-x) = -f(x)$$



Example: x, x^3, $\sin x$

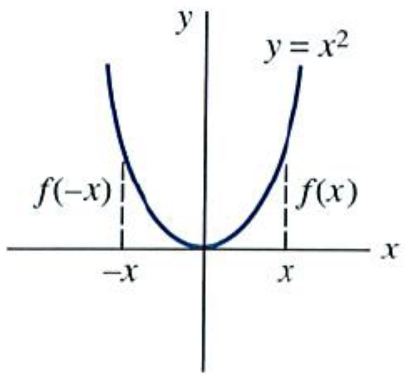
A function can be neither even nor odd. For example e^x, e^{-x}.

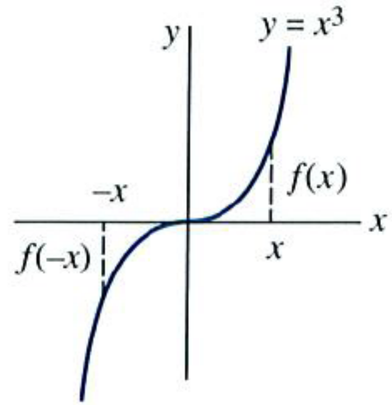
Theorem: Properties of even/odd functions

(a) The product of two even functions is even.
(b) The product of two odd functions is even.
(c) The product of an even function and an odd function is odd.

(d) The sum (difference) of two even functions is even.
(e) The sum (difference) of two odd functions is odd.

(f) If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.
(g) If f is odd, then $\int_{-a}^{a} f(x)dx = 0$.





Cosine and sine series

If f is an even function on $(-p, p)$ then the Fourier coefficients are

$$\begin{aligned}
a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)dx = \frac{2}{p}\int_{0}^{p} f(x)dx \\
a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx = \frac{2}{p}\int_{0}^{p} f(x)\cos\frac{n\pi}{p}x\,dx \\
b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx = 0
\end{aligned}$$

Similarly is f is odd on $(-p, p)$

$$\begin{aligned} a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)dx = 0 \\ a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx = 0 \\ b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx = \frac{2}{p}\int_{0}^{p} f(x)\sin\frac{n\pi}{p}x\,dx \end{aligned}$$

Definition: Fourier cosine and sine series

(i) The Fourier series of an even function on the interval $(-p, p)$ is the **cosine series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x \tag{1}$$

where

$$a_0 = \frac{2}{p} \int_0^p f(x) dx \tag{2}$$

$$a_n = \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi}{p} x \, dx \tag{3}$$

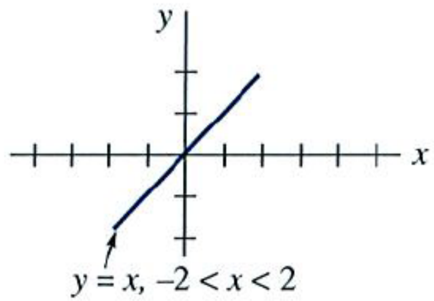
(ii) The Fourier series of an odd function on the interval $(-p, p)$ is the **sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x \tag{4}$$

where

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x \, dx \tag{5}$$

Example 1: Expansion in a sine series



Expand $f(x) = x$, $-2 < x < 2$ in a Fourier series.

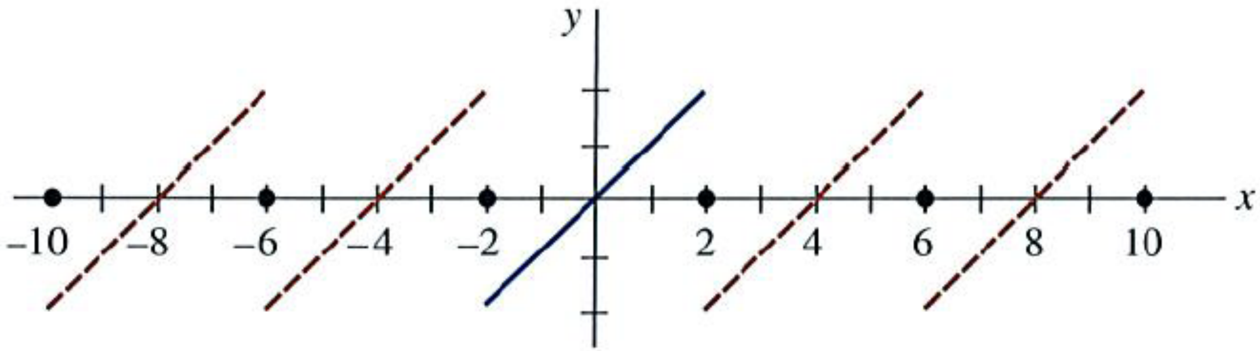
Solution: The given function is odd on the interval $(-2, 2)$, so we expand it in a sine series with $p = 2$. Using the integration by parts we get

$$b_n = \int_0^2 x \, \sin \frac{n\pi}{2} x \, dx = \frac{4(-1)^{n+1}}{n\pi}$$

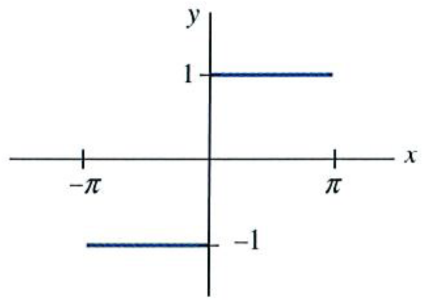
Therefore

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2} x$$

The series converges to the function on $(-2, 2)$ and the periodic extension (of period 4).



Example 2: Expansion in a sine series



$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \leq x < \pi \end{cases}$$

The function is odd on the interval $(-\pi, \pi)$. With $p = \pi$ we have

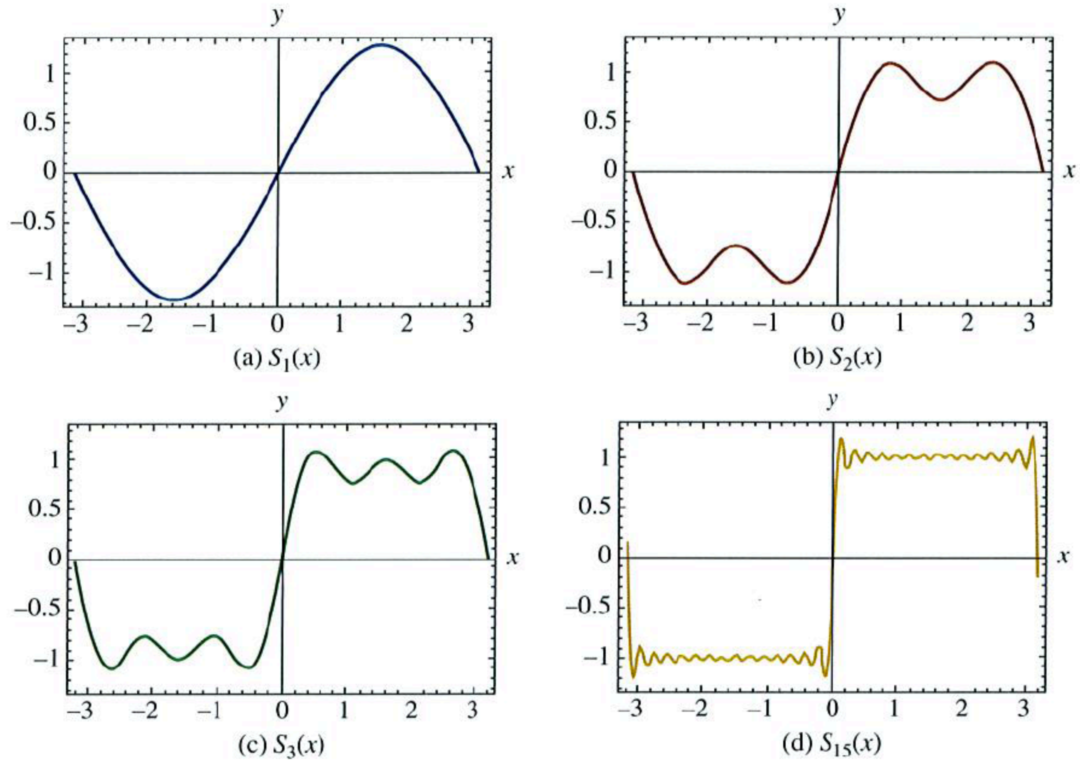
$$b_n = \frac{2}{\pi} \int_0^{\pi} (1)\ \sin nx\, dx = \frac{2}{\pi} \frac{1 - (-1)^n}{n}$$

and so

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin nx$$

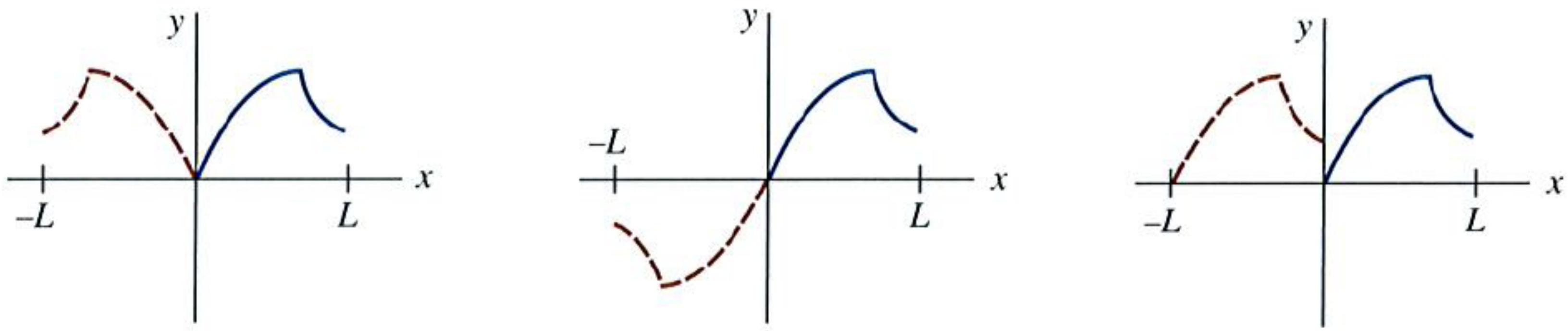
Gibbs phenomenon

If a function has discontinuities, then its Fourier series expansion exhibits ”overshooting” by the partial sums from the functional values near a point of discontinuity. This does not smooth out and remains fairly constant even for large N. This behavior of a Fourier series near a point of discontinuity at which f is discontinuous is known as the **Gibbs phenomenon**.



Half-range expansions

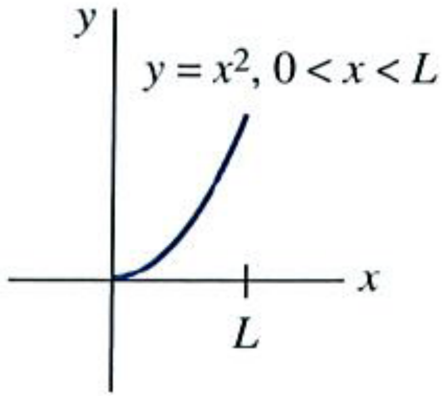
Note that the cosine and sine series utilize the definition of a function only on $0 < x < p$. So the cosine and sine expansions of a function defined on $(0, p)$ are **half-range expansions**. The choice of cosine or sine function determines whether the periodic extension of the function behaves as even or odd function on the full interval $(-p, p)$.



$f(x) = f(x + L)$

Example 3: Expansion in three series

Expand $f(x) = x^2$, $0 < x < L$ (a) in a cosine series, (b) in a sine series, (c) in a Fourier series.



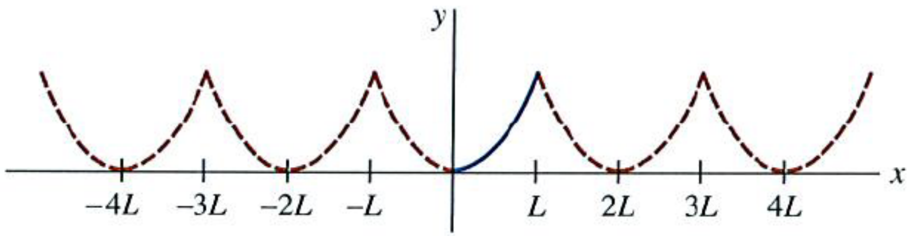
(a)

$$
\begin{aligned}
a_0 &= \frac{2}{L}\int_0^L x^2\,dx = \frac{2}{3}L^2 \\
a_n &= \frac{2}{L}\int_0^L x^2\,\cos\frac{n\pi}{L}x\,dx = \frac{4L^2(-1)^n}{n^2\pi^2}
\end{aligned}
$$

where the integration by parts was used twice to get a_n. Thus

$$
f(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi}{L}x
$$

The series converges to the $2L$-periodic even extension of f.



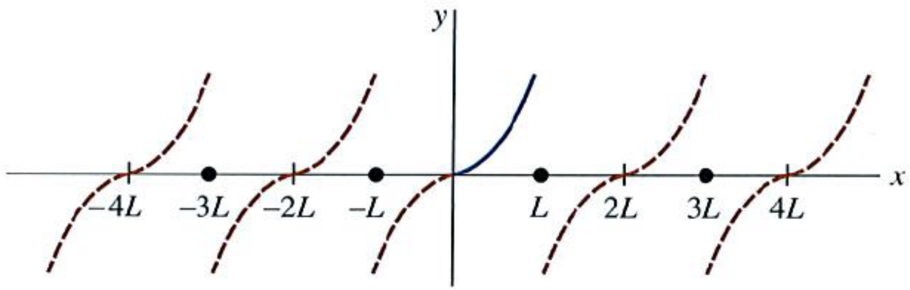
(a) Cosine series

(b)

$$b_n \;=\; \frac{2}{L}\int_0^L x^2 \;\sin\frac{n\pi}{L}x\,dx = \frac{2L^2(-1)^{n+1}}{n\pi} + \frac{4L^2}{n^3\pi^3}[(-1)^n - 1]$$

$$f(x) = \frac{2L^2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}}{n} + \frac{2}{n^3\pi^2}[(-1)^n - 1]\right\}\sin\frac{n\pi}{L}x$$

The series converges to the $2L$-periodic odd extension of f.



(b) Sine series

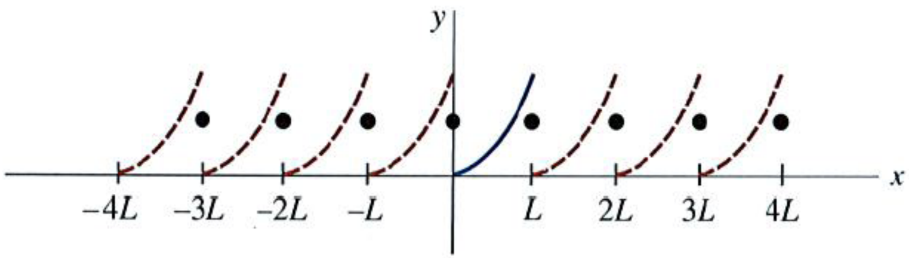
(c) with $p = L/2$, $1/p = 2/L$, and $n\pi/p = 2n\pi/L$, we have

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L x^2\,dx = \frac{2}{3}L^2 \\ a_n &= \frac{2}{L}\int_0^L x^2\,\cos\frac{n\pi}{L}x\,dx = \frac{L^2}{n^2\pi^2} \\ b_n &= \frac{2}{L}\int_0^L x^2\,\sin\frac{n\pi}{L}x\,dx = -\frac{L^2}{n\pi} \end{aligned}$$

Therefore

$$f(x) = \frac{L^2}{3} + \frac{L^2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{n^2\pi}\cos\frac{2n\pi}{L}x - \frac{1}{n}\sin\frac{2n\pi}{L}x\right\}$$

The series converges to the L-periodic extension of f.



(c) Fourier series

Periodic driving force

Fourier series can be useful in determining a particular solution of a DE describing a physical system in which the input or driving force is periodic.

Example 4:

We will find a particular solution of the DE describing an undamped spring/mass system

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + kx = f(t)$$

by first representing f by a half-range sine expansion and then assuming a particular solution of the form

$$x_p(t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{p}t$$

Consider the spring/mass system characterized by $m = 1/16\ kg$, and $k = 4\ N/m$ and driven by the periodic force defined as

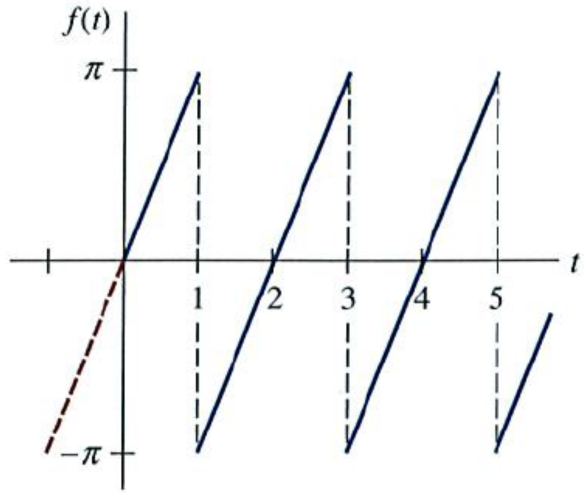
$$f(t) = \begin{cases} \pi t, & 0 < t < 1 \\ \pi(t-2) & 1 \le t < 2 \end{cases}$$

Note that we only need the half-range sine expansion of $f(t) = \pi t$, $0 < t < 1$. With $p = 1$ it follows using integration by parts that

$$b_n = 2\int_0^1 \pi t \sin n\pi t \, dt = \frac{2(-1)^{n+1}}{n}$$

The differential equation of motion is then

$$\frac{1}{16}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin n\pi t$$



To find a particular solution $x_p(t)$ we substitute

$$x_p(t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{p} t$$

into the equation and equate the coefficients of $\sin n\pi t$. This yields

$$x_p(t) = \sum_{n=1}^{\infty} \frac{32(-1)^{n+1}}{n(64 - n^2\pi^2)} \sin n\pi t$$

Observe that in the solution there is no integer $n \leq 1$ for which the denominator $64 - n^2\pi^2$ of B_n is zero. That means that in this system we will not observe the phenomenon of pure resonance.

In general, we observe pure resonance if there is a value of $n = N$ for which $N\pi/p = \omega$ where ω is the natural angular frequency of the system $\omega = \sqrt{k/m}$. In other words we have pure resonance if the Fourier expansion of the driving force $f(t)$ contains a term $\sin(N\pi/L)t$ or $\cos(N\pi/L)t$ that has the same frequency as the free vibrations.

Complex Fourier series

We will now recast the definition of the Fourier series into a **complex form** or **exponential form**.

A complex number can be written as $z = a + ib$ where a and b are real numbers and $i^2 = -1$. Also a complex number $\bar{z} = a - ib$ is the complex conjugate of the number z. Recall Euler's formula

$$e^{ix} = \cos x + i \sin x \qquad e^{-ix} = \cos x - i \sin x$$

Complex Fourier series

Using Euler's formulas above we can express the cosine and sine functions as

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Using these we can write the Fourier series of a function f as

$$\begin{aligned}
\frac{a_0}{2} &+ \sum_{n=1}^{\infty}\left[a_n \frac{e^{in\pi x/p} + e^{-in\pi x/p}}{2} + b_n \frac{e^{in\pi x/p} - e^{-in\pi x/p}}{2i}\right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{2}(a_n - ib_n)e^{in\pi x/p} + \frac{1}{2}(a_n + ib_n)e^{-in\pi x/p}\right] \\
&= c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/p} + \sum_{n=1}^{\infty} c_{-n} e^{-in\pi x/p}
\end{aligned}$$

where $c_0 = a_0/2$, $c_n = (a_n - ib_n)/2$, $c_{-n} = (a_n + ib_n)/2$.

When the function f is real, c_n and c_{-n} are complex conjugates and can also be written in terms of the complex exponential functions:

$$
\begin{aligned}
c_0 &= \frac{1}{2}\,\frac{1}{p}\int_{-p}^{p} f(x)\,dx \\
c_n &= \frac{1}{2}(a_n - ib_n) = \frac{1}{2}\left(\frac{1}{p}\int_{-p}^{p} f(x)\ \cos\frac{n\pi}{p}x\,dx - i\frac{1}{p}\int_{-p}^{p} f(x)\ \sin\frac{n\pi}{p}x\,dx\right) \\
&= \frac{1}{2p}\int_{-p}^{p} f(x)\left[\cos\frac{n\pi}{p}x - i\sin\frac{n\pi}{p}x\right]dx = \frac{1}{2p}\int_{-p}^{p} f(x)\,e^{-in\pi x/p}\,dx \\
c_{-n} &= \frac{1}{2}(a_n + ib_n) = \frac{1}{2}\left(\frac{1}{p}\int_{-p}^{p} f(x)\ \cos\frac{n\pi}{p}x\,dx + i\frac{1}{p}\int_{-p}^{p} f(x)\ \sin\frac{n\pi}{p}x\,dx\right) \\
&= \frac{1}{2p}\int_{-p}^{p} f(x)\left[\cos\frac{n\pi}{p}x + i\sin\frac{n\pi}{p}x\right]dx = \frac{1}{2p}\int_{-p}^{p} f(x)\,e^{in\pi x/p}\,dx
\end{aligned}
$$

Definition: Complex Fourier series

The **complex Fourier series** of functions f defined on an inteval $(-p, p)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{in\pi x/p}, \tag{6}$$

where

$$c_n = \frac{1}{2p} \int_{-p}^{p} f(x) \, e^{-in\pi x/p} dx, \qquad n = 0, \pm 1, \pm 2, ... \tag{7}$$

Example 1: Complex Fourier series

Expand $f(x) = e^{-x}$, $-\pi < x < \pi$ in a complex Fourier series.

Solution: With $p = \pi$, we get

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-x}\, e^{-inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-(in+1)x}\, dx \\ &= -\frac{1}{2\pi(in+1)}\left[e^{-(in+1)\pi} - e^{(in+1)\pi}\right] \end{aligned}$$

We can simplify the coefficients using Euler's formula

$$\begin{aligned} e^{-(in+1)\pi} &= e^{-\pi}(\cos n\pi - i\sin n\pi) = (-1)^n e^{-\pi} \\ e^{(in+1)\pi} &= e^{\pi}(\cos n\pi + i\sin n\pi) = (-1)^n e^{\pi} \end{aligned}$$

Since $\cos n\pi = (-1)^n$ and $\sin n\pi = 0$.

Hence

$$c_n = (-1)^n \frac{(e^{\pi} - e^{-\pi})}{2(in+1)\pi} = (-1)^n \, \frac{\sinh \pi}{\pi} \, \frac{1 - in}{n^2 + 1}$$

The complex Fourier series is then

$$f(x) = \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \, \frac{1 - in}{n^2 + 1} \, e^{inx}$$

The series converges to the 2π-periodic extension of f.

Fundamental frequency

The Fourier series in both the real and complex definitions define a periodic function and the **fundamental period** of that function (i.e. the periodic extension of f) is $T = 2p$. Since $p = T/2$ the definitions become respectively

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega x + b_n \sin n\omega x) \quad \text{and} \quad \sum_{n=-\infty}^{\infty} c_n e^{in\omega x}$$

where the number $\omega = 2\pi/T$ is called the **fundamental angular frequency**.

In Example 1, the periodic extension of the function has period $T = 2\pi$ and thus the fundamental angular frequency is $\omega = 2\pi/2\pi = 1$.

Frequency spectrum

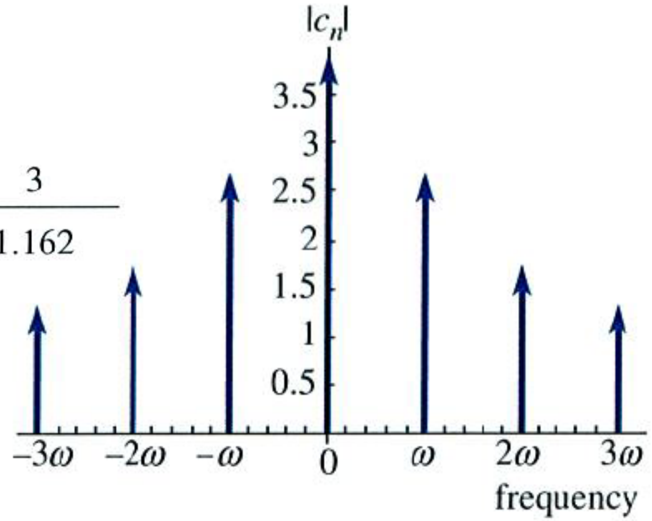
If f is periodic and has fundamental period T, the plot of the points $(n\omega, |c_n|)$, where ω is the fundamental angular frequency, and c_n are the coefficients of the complex Fourier series, is called the **frequency spectrum**.

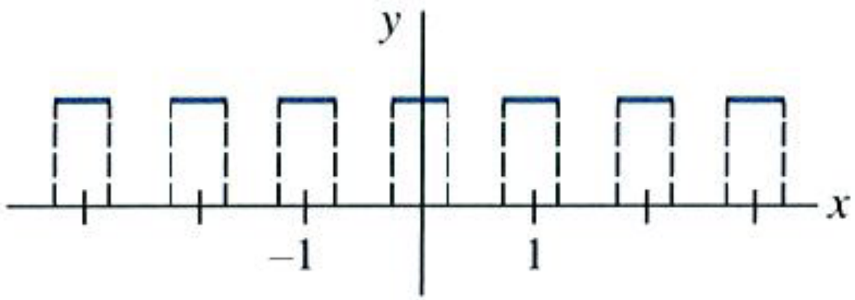
Example 2: Frequency spectrum

In Example 1, $\omega = 1$, so that $n\omega = 0, \pm 1, \pm 2, \ldots$. Using $|a + ib| = \sqrt{a^2 + b^2}$, we get

$$|c_n| = \frac{\sinh \pi}{\pi} \frac{1}{\sqrt{n^2 + 1}}$$

n	-3	-2	-1	0	1	2	3
$\lvert c_n \rvert$	1.162	1.644	2.599	3.676	2.599	1.644	1.162





Example 3: Frequency spectrum

$$f(x) = \begin{cases} 0, & -\frac{1}{2} < x < -\frac{1}{4} \\ 1, & -\frac{1}{4} < x < \frac{1}{4} \\ 0, & \frac{1}{4} < x < \frac{1}{2} \end{cases}$$

Here $T = 1 = 2p$, so $p = \frac{1}{2}$. The coefficients c_n are

$$\begin{aligned} c_n &= \int_{-1/2}^{1/2} f(x)\, e^{2in\pi x}\, dx = \int_{-1/4}^{1/4} 1\ .\ e^{2in\pi x}\, dx = \left[\frac{e^{2in\pi x}}{2in\pi}\right]_{-1/4}^{1/4} \\ &= \frac{1}{n\pi}\frac{e^{in\pi/2} - e^{-in\pi/2}}{2i} = \frac{1}{n\pi}\sin\frac{n\pi}{2} \end{aligned}$$

Since this result is not valid for $n = 0$, we compute c_0 separately: $c_0 = \int_{-1/4}^{1/4} dx = \frac{1}{2}$

The following table shows some of the values of $|c_n|$:

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\lvert c_n \rvert$	$\frac{1}{5\pi}$	0	$\frac{1}{3\pi}$	0	$\frac{1}{\pi}$	$\frac{1}{2}$	$\frac{1}{\pi}$	0	$\frac{1}{3\pi}$	0	$\frac{1}{5\pi}$