Additional operational properties

How to find the Laplace transform of a function $f(t)$ that is multiplied by a monomial t^n, the transform of a special type of integral, and the transform of a periodic function?

Multiplying a function by t^n

$$\frac{\mathrm{d}}{\mathrm{d}s}F(s) = \frac{\mathrm{d}}{\mathrm{d}s}\int_0^{\infty} e^{-st} f(t)\, dt = \int_0^{\infty} \frac{\partial}{\partial s}\left[e^{-st} f(t)\right]\, dt = -\int_0^{\infty} e^{-st}\, t\, f(t)\, dt = -\mathcal{L}\{t\, f(t)\}$$

that is

$$\mathcal{L}\{t\, f(t)\} = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{f(t)\}$$

Similarly

$$\mathcal{L}\left\{t^2\, f(t)\right\} = \mathcal{L}\{t.t\, f(t)\} = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{t\, f(t)\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(-\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{f(t)\}\right) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{L}\{f(t)\}$$

Theorem: Derivatives of transforms

If $F(s) = \mathcal{L}\{f(t)\}$ and $n = 1, 2, 3, \ldots$ then

$$\mathcal{L}\{t^n \ f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s) \tag{1}$$

Example 1: $\mathcal{L}\{t \ \sin kt\}$

With $f(t) = \sin kt$, $F(s) = k/(s^2 + k^2)$, and $n = 1$, the theorem above gives

$$\mathcal{L}\{t \ \sin kt\} = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{\sin kt\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{k}{s^2 + k^2}\right) = \frac{2ks}{(s^2 + k^2)^2}$$

Evaluate $\mathcal{L}\left\{t^2 \ \sin kt\right\}$ and $\mathcal{L}\left\{t^3 \ \sin kt\right\}$.

Example 2: $\quad x'' + 16x = \cos 4t, \qquad x(0) = 0, \quad x'(0) = 1$

The Laplace transform of the DE gives

$$(s^2 + 16)X(s) = 1 + \frac{s}{s^2 + 16}$$
$$X(s) = \frac{1}{s^2 + 16} + \frac{s}{(s^2 + 16)^2}$$

In the example 1 we have got

$$\mathcal{L}^{-1}\left\{\frac{2ks}{(s^2 + k^2)^2}\right\} = t \ \sin kt$$

and so with the identification $k = 4$, we obtain

$$x(t) = \frac{1}{4}\,\mathcal{L}^{-1}\left\{\frac{4}{s^2 + 16}\right\} + \frac{1}{8}\,\mathcal{L}^{-1}\left\{\frac{8s}{(s^2 + 16)^2}\right\} = \frac{1}{4}\sin 4t + \frac{1}{8}\,t \ \sin 4t$$

Transforms of Integrals

Convolution

If functions f and g are piecewise continuous on $[0, \infty)$ then a special product, $f * g$, defined by the integral

$$f * g = \int_0^t f(\tau)g(t-\tau)d\tau \qquad (2)$$

is called the convolution of f and g. The convolution $f * g$ is a function of t.

Example:

$$e^t * \sin t = \int_0^t e^\tau \sin(t-\tau)\, d\tau = \frac{1}{2}\left(-\sin t - \cos t + e^t\right)$$

The convolution of two functions is commutative:

$$f * g = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau = g * f \tag{3}$$

The Laplace transform of the convolution $f * g$ is the product of Laplace transforms of f and g, thus we can find the Laplace transform of the convolution without performing the convolution integral.

Convolution theorem

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\} = F(s)G(s). \tag{4}$$

The Proof: see D.G. Zill et al., Advanced Engineering Mathematics, 4th Ed., p.222.

Example 3

$$\mathcal{L}\left\{e^t * \sin t\right\} = \mathcal{L}\left\{\int_0^t e^\tau \sin(t-\tau)\, d\tau\right\} = \mathcal{L}\left\{e^t\right\}\mathcal{L}\{\sin t\} = \frac{1}{s-1}.\frac{1}{s^2+1} = \frac{1}{(s-1)(s^2+1)}$$

Inverse form of the convolution theorem

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g \tag{5}$$

Example 4: $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\}$

Let

$$F(s) = G(s) = \frac{1}{s^2 + k^2}$$

then

$$f(t) = g(t) = \frac{1}{k}\mathcal{L}^{-1}\left\{\frac{k}{s^2 + k^2}\right\} = \frac{1}{k}\sin kt$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2 + k^2)^2}\right\} = \frac{1}{k^2}\int_0^t \sin k\tau \sin k(t - \tau) d\tau$$

Using

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)] \tag{6}$$

with $A = k\tau$ and $B = k(t-\tau)$, we can carry out the integration

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} &= \frac{1}{k^2}\int_0^t \sin k\tau \sin k(t-\tau)d\tau \\
&= \frac{1}{2k^2}\int_0^t [\cos k(2\tau - t) - \cos kt]d\tau \\
&= \frac{1}{2k^2}\left[\frac{1}{2k}\sin k(2\tau - t) - \tau\cos kt\right]_0^t \\
&= \frac{\sin kt - kt\cos kt}{2k^3}
\end{aligned}$$

Transform of an integral

When $g(t) = 1$ and $\mathcal{L}\{g(t)\} = G(s) = 1/s$, the convolution theorem implies that the Laplace transform of the integral of f is

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s} \tag{7}$$

The inverse form of the equation above

$$\int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} \tag{8}$$

can be used instead of partial fractions when s^n is a factor of the denominator and $f(t) = \mathcal{L}^{-1}\{F(s)\}$ is easy to integrate.

Examples:

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1/(s^2+1)}{s}\right\} = \int_0^t \sin\tau \, d\tau = 1 - \cos t \\
\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1/s(s^2+1)}{s}\right\} = \int_0^t (1 - \cos\tau) \, d\tau = t - \sin t \\
\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1/s^2(s^2+1)}{s}\right\} = \int_0^t (\tau - \sin\tau) \, d\tau = \frac{1}{2}t^2 - 1 + \cos t
\end{aligned}
$$

and so on.

Volterra integral equation

The convolution theorem and the Laplace transform of integrals are useful for solving types of equations in which an unknown function appears under an integral sign.

For example, we can solve a Volterra integral equation for $f(t)$

$$f(t) = g(t) + \int_0^t f(\tau)h(t-\tau)\,d\tau \tag{9}$$

The functions $g(t)$ and $h(t)$ are known. Notice that the integral above is of the convolution form.

Example 5: An integral equation

Solve

$$f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)\, e^{t-\tau}\, d\tau$$

for $f(t)$.

We identify $h(t-\tau) = e^{t-\tau}$, so $h(t) = e^t$ and take the Laplace transform of each term; in particular the integral transforms as the product of $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\left\{e^t\right\} = 1/(s-1)$:

$$F(s) = 3.\frac{2}{s^3} - \frac{1}{s+1} - F(s).\frac{1}{s-1}$$

After solving the equation for $F(s)$ and carrying the partial fraction decomposition, we get

$$F(s) = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}$$

the inverse Laplace transform then gives

$$\begin{aligned} f(t) &= 3\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} - \mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ &= 3t^2 - t^3 + 1 - 2e^{-t} \end{aligned}$$

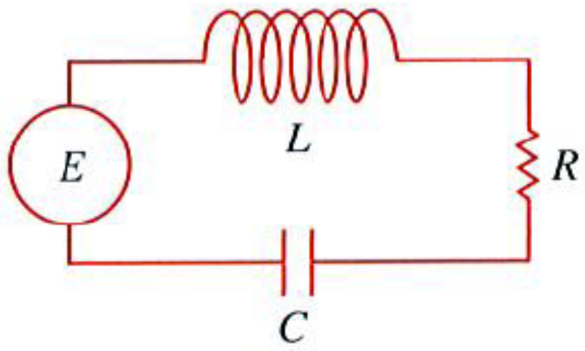
Series circuits

Recall that in a single-loop or series circuits, Kirchhoff's second law states that the sum of the voltage drops across an inductor, resistor, and capacitor is equal to the impressed voltage $E(t)$. The voltage drops across an inductor, resistor and capacitor are, respectively,

$$L\frac{\mathrm{d}i}{\mathrm{d}t}, \qquad R\, i(t), \qquad \text{and} \quad \frac{1}{C}\int_0^t i(\tau)\, d\tau$$

where $i(t)$ is the current, and L, R, and C are constants. It follows that the current in a circuit is govern by the **integrodifferential equation**

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + R\, i(t) + \frac{1}{C}\int_0^t i(\tau)\, d\tau = E(t)$$



Example 6: An integrodifferential equation

Determine the current $i(t)$ in a single-loop LRC-circuit when $L = 0.1\ h$, $R = 2\ \Omega$, $C = 0.1\ f$, $i(0) = 0$, and the impressed voltage is

$$E(t) = 120\,t - 120\,t\,\mathcal{U}(t-1)$$

Using the data above, the integrodifferential equation we are to solve is

$$0.1\frac{\mathrm{d}i}{\mathrm{d}t} + 2\,i(t) + 10\int_0^t i(\tau)\,d\tau = 120\,t - 120\,t\,\mathcal{U}(t-1)$$

Using $\mathcal{L}\left\{\int_0^t i(\tau)\,d\tau\right\} = I(s)/s$, where $I(s) = \mathcal{L}\{i(t)\}$, the Laplace transform of the equation is

$$0.1\ s\,I(s) + 2\,I(s) + 10\frac{I(s)}{s} = 120\left[\frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}\right]$$

where we have used $\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$ (see the section on the second translation theorem). Solving the equation above for $I(s)$ yields

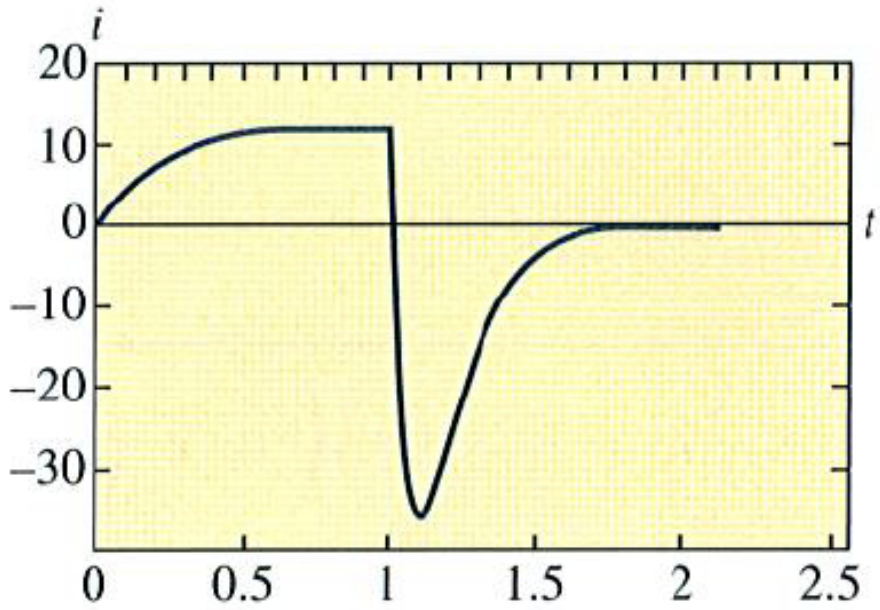
$$\begin{aligned} I(s) &= 1200\left[\frac{1}{s(s+10)^2} - \frac{1}{s(s+10)^2}e^{-s} - \frac{1}{(s+10)^2}e^{-s}\right] \\ &= 1200\left[\frac{1/100}{s} - \frac{1/100}{s+10} - \frac{1/10}{(s+10)^2} - \frac{1/100}{s}e^{-s}\right. \\ &\qquad \left. + \frac{1/100}{s+10}e^{-s} + \frac{1/10}{(s+10)^2}e^{-s} - \frac{1}{(s+10)^2}e^{-s}\right] \end{aligned}$$

From the inverse form of the second translation theorem, we obtain the current

$$\begin{aligned} i(t) &= 12\,[1 - \mathcal{U}(t-1)] - 12\left[e^{-10t} - e^{-10(t-1)}\mathcal{U}(t-1)\right] \\ &\quad -120te^{-10t} - 1080(t-1)e^{-10(t-1)}\,\mathcal{U}(t-1) \end{aligned}$$

which can be rewritten as the piecewise-defined function

$$i(t) = \begin{cases} 12 - 12e^{-10t} - 120te^{-10t}, & 0 \le t < 1 \\ -12e^{-10t} + 12e^{-10(t-1)} - 120e^{-10t} - 1080(t-1)e^{-10(t-1)}, & t \ge 1. \end{cases}$$



Transform of a periodic function

Periodic function

If a periodic function f has period T, $T > 0$, then $f(t + T) = f(t)$.

Theorem: Transform of a periodic function

If $f(t)$ is a piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period T, the

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

The Proof: see D.G. Zill et al., Advanced Engineering Mathematics, 4th Ed., p.226.

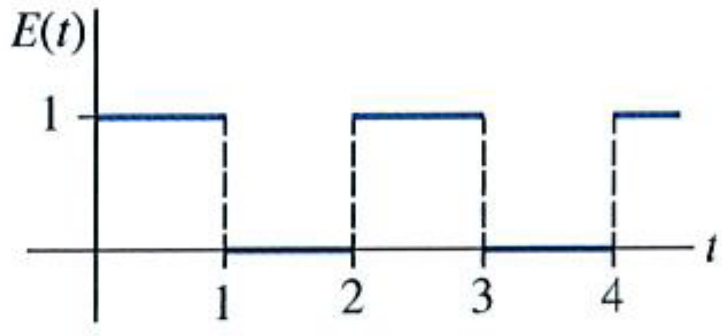
Example 7: Transform of a periodic function

Find the Laplace transform of the square wave $E(t)$ with the period $T = 2$. For $0 \leq t < 2$, it can be defined by

$$E(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & 1 \leq t < 2. \end{cases}$$

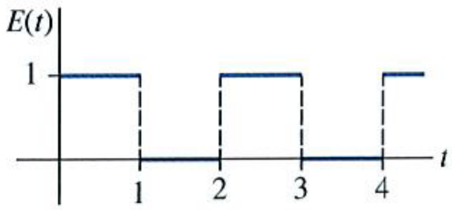
and outside of the interval by $f(t+2) = f(t)$. From the theorem above, we have

$$\begin{aligned} \mathcal{L}\{E(t)\} &= \frac{1}{1-e^{-2s}} \int_0^2 e^{-st}\, E(t)\, dt = \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st}\, 1\, dt + \int_1^2 e^{-st}\, 0\, dt \right] \\ &= \frac{1}{1-e^{-2s}} \frac{1-e^{-s}}{s} = \frac{1}{(1+e^{-s})(1-e^{-s})} \frac{1-e^{-s}}{s} \\ &= \frac{1}{s(1+e^{-s})} \end{aligned}$$



Example 8: A periodic impressed voltage

Determine the current $i(t)$ in a single-loop LR-series circuit



$$L\frac{\mathrm{d}i}{\mathrm{d}t} + R\,i = E(t)$$

where $E(t)$ is the same function that we studied in the previous example.

The Laplace transform of the differential equation is

$$L\,s\,I(s) + R\,I(s) = \frac{1}{s\,(1+e^{-s})}$$

or

$$I(s) = \frac{1/L}{s(s+R/L)}\,\frac{1}{1+e^{-s}}$$

To find the inverse Laplace transform of the last expression, we identify $x = e^{-s}$, $s > 0$, and use the geometric series

$$
\begin{aligned}
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + \ldots \\
\frac{1}{1+e^{-s}} &= 1 - e^{-s} + e^{-2s} - e^{-3s} + \ldots
\end{aligned}
$$

From the partial fraction decomposition

$$\frac{1}{s(s+R/L)} = \frac{L/R}{s} - \frac{L/R}{s+R/L}$$

we can rewrite $I(s)$ as

$$
\begin{aligned}
I(s) &= \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s+R/L}\right)\left(1 - e^{-s} + e^{-2s} - e^{-3s} + \ldots\right) \\
&= \frac{1}{R}\left(\frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \ldots\right) - \frac{1}{R}\left(\frac{1}{s+R/L} - \frac{e^{-s}}{s+R/L} + \frac{e^{-2s}}{s+R/L} - \frac{e^{-3s}}{s+R/L} + \ldots\right)
\end{aligned}
$$

By applying the second translation theorem to each term of both series we get

$$\begin{aligned} i(t) = \quad & \frac{1}{R}\left(1 - \mathcal{U}(t-1) + \mathcal{U}(t-2) - \mathcal{U}(t-3) + \ldots\right) \\ - \quad & \frac{1}{R}\left(e^{-Rt/L} - e^{-R(t-1)/L}\,\mathcal{U}(t-1) + e^{-R(t-2)/L}\,\mathcal{U}(t-2) - e^{-R(t-3)/L}\,\mathcal{U}(t-3) + \ldots\right) \end{aligned}$$

or equivalently

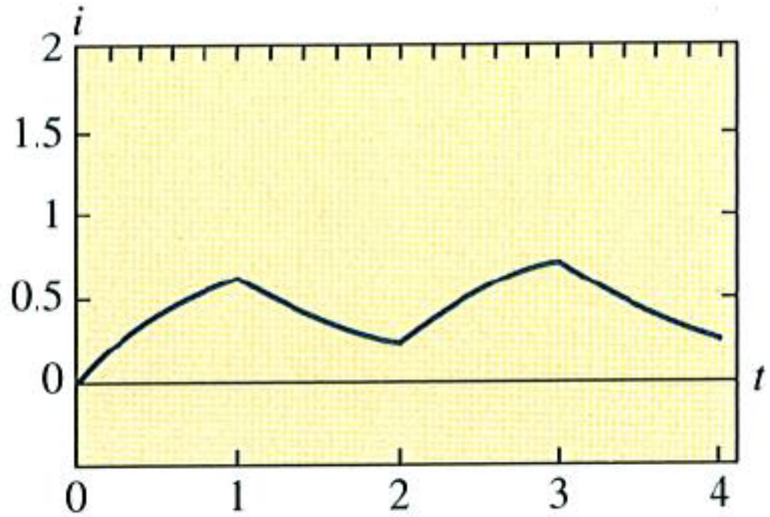
$$i(t) = \frac{1}{R}\left(1 - e^{-Rt/L}\right) + \frac{1}{R}\sum_{n=1}^{\infty}(-1)^n\left(1 - e^{-R(t-n)/L}\right)\mathcal{U}(t-n)$$

To interpret the solution, let us assume for illustration that $R = 1$, $L = 1$ and $0 \le t < 4$. We get

$$i(t) = 1 - e^{-t} - \left(1 - e^{-(t-1)}\right)\mathcal{U}(t-1) + \left(1 - e^{-(t-2)}\right)\mathcal{U}(t-2) - \left(1 - e^{-(t-3)}\right)\mathcal{U}(t-3)$$

or in other words

$$i(t) = \begin{cases} 1 - e^{-t}, & 0 \le t < 1 \\ -e^{-t} + e^{-(t-1)}, & 1 \le t < 2 \\ 1 - e^{-t} + e^{-(t-1)} - e^{-(t-2)}, & 2 \le t < 3 \\ -e^{-t} + e^{-(t-1)} - e^{-(t-2)} + e^{-(t-3)}, & 3 \le t < 4 \end{cases}$$



The Dirac delta function

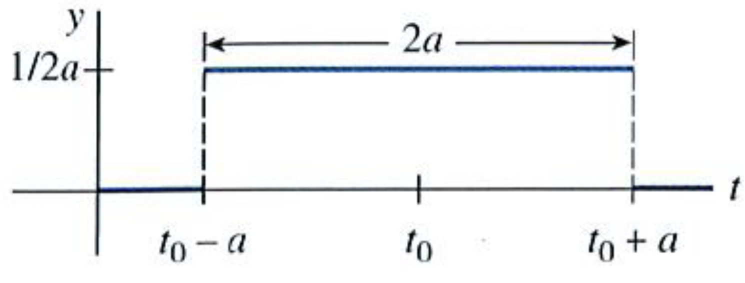
Unit pulse

$$\delta_a(t-t_0) = \begin{cases} 0, & 0 \le t < t_0 - a \\ \frac{1}{2a}, & t_0 - a \le t < t_0 + a \\ 0, & t \ge t_0 + a \end{cases}$$

$a > 0$, $t_0 > 0$.

For a small value of a, $\delta_a(t-t_0)$ is essentially a constant function of large magnitude that is ”on” for just a very short period of time, around t_0. As $a \to 0$ the magnitude of the unit pulse grows to ∞ while the area under the unit pulse is constant and equals to 1:

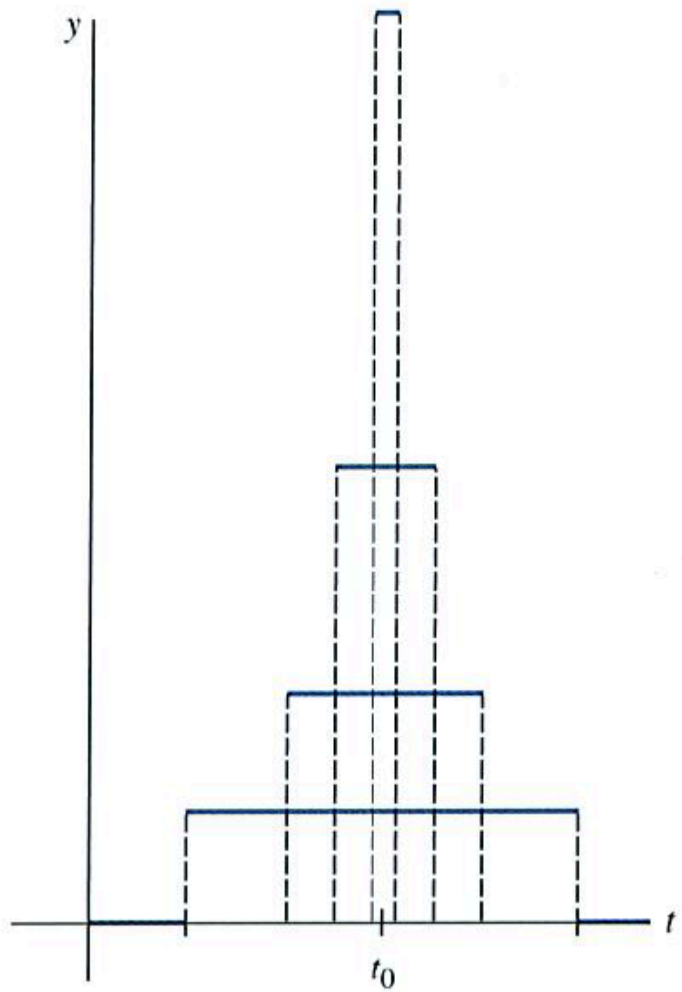
$$\int_0^\infty \delta_a(t-t_0)\, dt = 1$$



The Dirac delta ”function”

is defined as the limit of the unit pulse

$$\delta(t - t_0) = \lim_{a \to 0} \delta_a(t - t_0)$$



and it can be characterized by two properties:

(i)

$$\delta(t - t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \neq t_0 \end{cases}$$

(ii)

$$\int_0^{\infty} \delta(t - t_0)\, dt = 1.$$

It is possible to obtain the Laplace transform of the Dirac delta function by the formal assumption that $\mathcal{L}\{\delta(t-t_0)\} = \lim_{a\to 0} \mathcal{L}\{\delta_a(t-t_0)\}$.

Theoremn: Transform of the Dirac delta function

For $t_0 > 0$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0} \tag{12}$$

<u>Proof:</u>

$$\begin{aligned}
\delta_a(t-t_0) &= \frac{1}{2a}\left[\mathcal{U}(t-(t_0-a)) - \mathcal{U}(t-(t_0+a))\right] \\
\mathcal{L}\{\delta_a(t-t_0)\} &= \frac{1}{2a}\left[\frac{e^{-s(t_0-a)}}{s} - \frac{e^{-s(t_0+a)}}{s}\right] = e^{-st_0}\left(\frac{e^{sa}-e^{-sa}}{2sa}\right) \\
\mathcal{L}\{\delta(t-t_0)\} &= \lim_{a\to 0}\mathcal{L}\{\delta_a(t-t_0)\} = e^{-st_0}\lim_{a\to 0}\left(\frac{e^{sa}-e^{-sa}}{2sa}\right) = e^{-st_0}
\end{aligned}$$

Now, when $t_0 = 0$, $\mathcal{L}\{\delta(t)\} = 1$. This emphasizes the fact that the delta ”function” is not a usual type of function, since we expect $\mathcal{L}\{f(t)\} \to 0$ as $s \to \infty$.

Example: Two initial value problems (the delta-function kicked oscillator)
Solve $y'' + y = 4\delta(t - 2\pi)$ subject to **(a)** $y(0) = 1$, $y'(0) = 0$, and **(b)** $y(0) = 0$, $y'(0) = 0$.

(a) The Laplace transform of the DE is

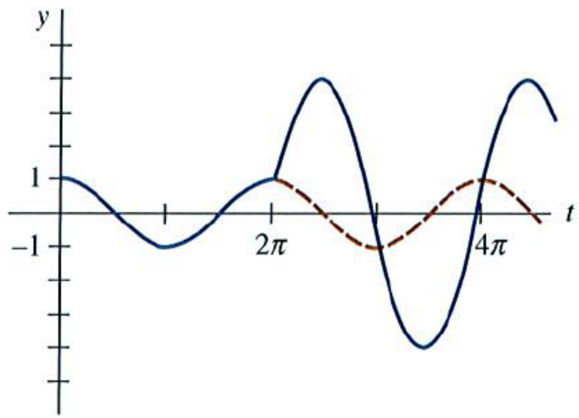
$$s^2 Y(s) - s + Y(s) = 4e^{-2\pi s} \qquad \Rightarrow \qquad Y(s) = \frac{s}{s^2+1} + \frac{4e^{-2\pi s}}{s^2+1}$$

Using the inverse form of the second translation theorem, we get

$$y(t) = \cos t + 4\sin(t - 2\pi)\mathcal{U}(t - 2\pi)$$

and since $\sin(t - 2\pi) = \sin t$, the solution can be rewritten as

$$y(t) = \begin{cases} \cos t, & 0 \le t < 2\pi \\ \cos t + 4\sin t, & t \ge 2\pi. \end{cases}$$



(b) The transform of the DE in this case is simpler

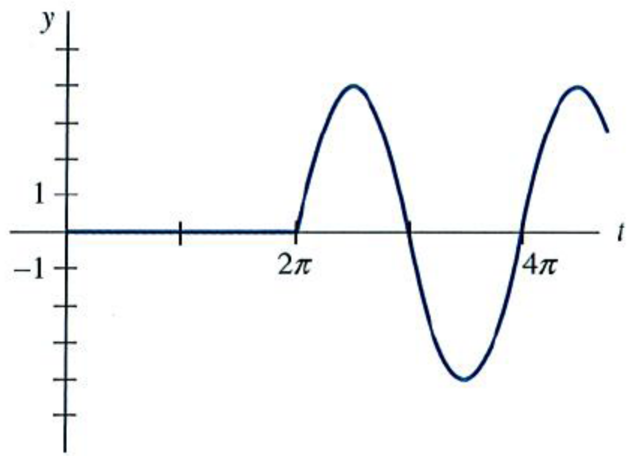
$$Y(s) = \frac{4e^{-2\pi s}}{s^2 + 1}$$

and so the final result is

$$y(t) = 4\sin(t - 2\pi)\mathcal{U}(t - 2\pi)$$

or

$$y(t) = \begin{cases} 0, & 0 \le t < 2\pi \\ 4\sin t, & t \ge 2\pi. \end{cases}$$



Remarks:

(i) The delta function is not a proper function but rather an example of distribution. It is best characterized by its effect on the other functions: if f is a continuous function then

$$\int_0^{\infty} f(t)\, \delta(t - t_0)\, dt = f(t_0)$$

This is known as the **sifting** property.

(ii) Recall that the transfer function of a general nth-order DE with constant coefficients is $W(s) = 1/P(s)$ where $P(s) = a_n s^n + a_{n-1} s^{n-1} + ... + a_0$. The transfer function is the Laplace transform of function $w(t)$ called the **weight function** of a linear system.

How does this work if the input is impulsive, i.e. it has the form of the delta function?

Consider the second-order linear system in which the input is the unit impulse at $t = 0$

$$a_2 y'' + a_1 y' + a_0 y = \delta(t), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Applying the Laplace transform and using $\mathcal{L}\{\delta(t)\} = 1$ shows that the transform of the response y in this case is the transfer function

$$Y(s) = \frac{1}{a_2 s^2 + a_1 s + a_0} = \frac{1}{P(s)} = W(s)$$

and so

$$y = \mathcal{L}^{-1}\left\{\frac{1}{P(s)}\right\} = w(t)$$

Thus, in general, the weight function $y = w(t)$ of an nth-order linear system is the zero-state response of the system to a unit pulse, and for this reason $w(t)$ is called the **impulse response** of the system.