HIGHER ORDER DIFFERENTIAL EQUATIONS

Theory of linear equations

Initial-value and boundary-value problem

nth-order initial value problem is

Solve:
$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Subject to:
$$y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots, y^{(n-1)} = y_{n-1} \qquad (1)$$

we seek a function defined on an interval I, containing x_0, that satisfies the DE and the n initial conditions above.

Higher-order ODEs: overview

- General aspects
- Reduction of order
- Homogeneous linear equation with constant coefficients
- Method of undetermined coefficients
- Method of variation of parameters
- Linear models
- Examples

Existence and uniqueness

Theorem: Existence of a unique solution

Let $a_n(x)$, $a_{n-1}(x)$, ... , $a_1(x)$, $a_0(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ in any point in this interval, then a solution $y(x)$ of the initial value problem (1) exists on the interval and is unique.

Example: Unique solution of an IVP

$$3y''' + 5y'' - y' + 7y = 0, \qquad y(1) = 0, y'(1) = 0, y''(1) = 0$$

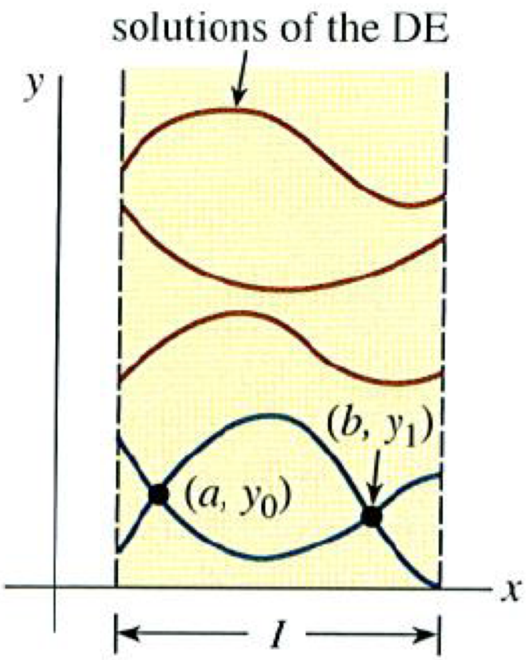
has the trivial solution $y = 0$. Since the DE is linear with constant coefficients, all the conditions of the theorem are fulfilled, and thus $y = 0$ is the *only* solution on any interval containing $x = 1$.

Boundary-value problem

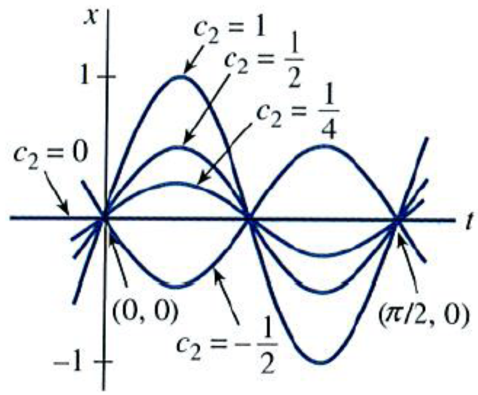
consists of solving a linear DE of order two or greater in which the dependent variable y or its derivatives are specified at *different points*. Example: a two-point BVP

Solve: $$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Subject to boundary conditions: $$y(x_0) = y_0,\ y(b) = y_1 \qquad (2)$$



A BVP can have many, one or no solutions:



The DE $x'' + 16x = 0$ has the two-parameter family of solutions $x = c_1 \cos 4t + c_2 \sin 4t$. Consider the BVPs:

(1) $x(0) = 0$, and $x(\pi/2) = 0 \Rightarrow c_1 = 0$ and the solution satisfies the DE for any value of c_2, thus the solution of this BVP is the one-parameter family $x = c_2 \sin 4t$.

(2) $x(0) = 0$, and $x(\pi/8) = 0 \Rightarrow c_1 = 0$ and $c_2 = 0$, so the only solution to this BVP is $x = 0$.

(3) $x(0) = 0 \Rightarrow c_1 = 0$ again but the second condition $x(\pi/2) = 1$ leads to the contradiction: $1 = c_2 \sin 2\pi = c_2.0 = 0$.

Homogeneous equations

nth-order **homogeneous** differential equation

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ \ldots\ + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = 0 \tag{3}$$

nth-order **nonhomogeneous** differential equation ($g(x) \neq 0$)

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ \ldots\ + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x) \tag{4}$$

Examples:

(1) Homogeneous DE: $2y'' + 3y' - 5y = 0$

(2) Nonhomogeneous DE: $x^2y''' + 6y' + 10y = e^x$.

To solve a nonhomogeneous DE, we must first be able to solve the **associated homogeneous equation**.

We will soon proceed to the general theory of nth-order linear equations which we will present through a number of definitions and theorems. To avoid needless repetition, we make (and remeber) the following assumptions:

on some common interval I

- the coefficients $a_i(x)$, $i = 0, 1, 2, ..., n$ are continuous;
- the function $g(x)$ on r. h. s. is continuous; and
- $a_n(x) \neq 0$ for every x in the interval.

Differential operators

Examples:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}y = Dy \quad \text{or} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = D\,(Dy) = D^2y \quad \text{and in general} \quad \frac{\mathrm{d}^ny}{\mathrm{d}x^n} = D^ny$$

***n*th-order differential operator:**

polynomial expressions involving D are also differential operators

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \ \ldots \ + a_1(x)D + a_0(x)$$

An nth-order differential operator is a **linear operator**, that is, it satisfies

$$L\,[\alpha f(x) + \beta g(x)] = \alpha L\,(f(x)) + \beta L\,(g(x)) \tag{5}$$

Differential equations

Any linear differential equation can be expressed in terms of the D notation.

Example

$$\begin{aligned} y'' + 5y' + 6y &= 5x - 3 \\ D^2y + 5Dy + 6y &= 5x - 3 \\ (D^2 + 5D + 6)y &= 5x - 3 \end{aligned}$$

The nth-order linear differential equations can be written compactly as

Homogeneous:	$L(y) = 0$
Non-homogeneous:	$L(y) = g(x)$

Superposition principle

Theorem: Superposition principle - homogeneous equations

Let y_1, y_2, ... , y_k be solutions of the homogeneous nth-order DE (3) on an interval I, then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_k y_k(x) = \sum_{i=1}^{k} c_i y_i(x),$$

where the c_i, $i = 1, 2, ..., k$ are arbitrary constants, is also a trivial solution.

Proof: The case $k = 2$. Let $y_1(x)$ and $y_2(x)$ be solutions of $L(y) = 0$, then also

$$L(y) = L\left[c_1 y_1(x) + c_2 y_2(x)\right] = c_1 L(y_1) + c_2 L(y_2) = 0$$

Corollaries

(a) A constant multiple $y = c_1 y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear DE is also a solution.

(b) A homogeneous linear DE always possesses the trivial solution $y = 0$.

Example: Superposition - homogeneous DE
Let $y_1 = x^2$ and $y_2 = x^2 \ln x$ be both solutions of the homogeneous linear DE $x^3 y''' - 2xy' + 4y = 0$ on the interval $I = (0, \infty)$.

Show that by superposition principle, the linear combination

$$y = c_1 x^2 + c_2 x^2 \ln x$$

is also a solution of the equation on the interval.

Linear dependence and linear independence

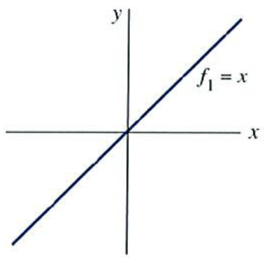
Definition:
A set of functions $f_1(x), f_2(x), ..., f_n(x)$ is said to be **linearly dependent** on an interval I if there exist constants $c_1, c_2, ..., c_n$, not all zero, s.t.

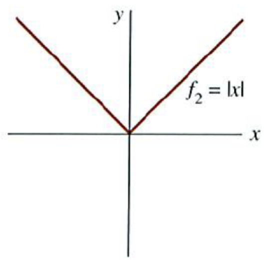
$$c_1 f_1(x) + c_2 f_2(x) + ... + c_n f_n(x) = 0 \tag{6}$$

for every x in the interval. If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent**.

Example: If two functions are linearly dependent, then one is simply a constant multiple of the other: assuming $c_1 \neq 0$, $c_1 f_1(x) + c_2 f_2(x) = 0 \Rightarrow f_1(x) = -(c_2/c_1) f_2(x)$. For example $f_1(x) = \sin(x)\cos(x)$ and $f_2(x) = \sin(2x) = 2f_1(x)$.

Two functions are linearly independent when neither is a constant multiple of the other on an interval. For example $f_1(x) = x$ and $f_2(x) = |x|$ on $I = (-\infty, \infty)$.





Solutions of differential equations

We are primarily interested in linearly independent solutions of linear DEs.

How to decide whether n solutions $y_1, y_2, ..., y_n$ of a homogeneous linear nth-order DE (3) are linearly independent?

Definition: Wronskian

Suppose each of the functions $f_1(x), f_2(x), ..., f_n(x)$ possesses at least $n-1$ derivatives. The determinant

$$W(f_1, f_2, ..., f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the **Wronskian** of the functions.

Theorem: Criterion for linearly independent solutions

Let $y_1, y_2, ..., y_n$ be n solutions of the homogeneous linear nth-order DE (3) on an interval I. Then the set of solutions is **linearly independent** on I if and only if $W(y_1, y_2, ..., y_n) \neq 0$ for every x in the interval.

Definition: Fundamental set of solutions

Any set $y_1, y_2, ..., y_n$ of n linearly independent solutions of the homogeneous linear nth-order DE (3) on an interval I is said to be a **fundamental set of solutions** on the interval.

Theorem: Existence of a fundamental set

There exists a fundamental set of solutions for the homogeneous linear nth-order DE (3) on an interval I.

Theorem: General solution - homogeneous equations

Let $y_1, y_2, ..., y_n$ be a fundamental set of solutions of the homogeneous linear nth-order DE (3) on an interval I. Then the **general** solution of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x)$$

where c_i, $i = 1, 2, ..., n$ are arbitrary constants.
For proof for the case $n = 2$ see D.G. Zill et al., Advanced Engineering Mathematics, 4th Edition, p. 104.

Example 1:

The functions $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are both solutions of the homogeneous linear DE $y'' - 9y = 0$ on $(-\infty, \infty)$.

Calculate the Wronskian and determine whether the functions form a fundamental set of solutions. If yes, determine a general solution.

Example 2:

The function $y = 4\sinh 3x - 5e^{3x}$ is a solution of the DE in Example 1 above. Verify this.

We must be able to obtain this solution from the general solution $y = c_1e^{3x} + c_2e^{-3x}$. What values the constants c_1 and c_2 have to have to get the solution above.

Example 3:

The functions $y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$ satisfy the third order DE $y''' - 6y'' + 11y' - 6y = 0$. Determine whether these functions form the fundamental set of solutions on $(-\infty, \infty)$, and write down the general solution.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{4}$$

Nonhomogeneous equations

Any function y_p free of any arbitrary parameters that satisfies (4) is said to be a **particular solution** of the equation.

For example, $y_p = 3$ is a particular solution of $y'' + 9y = 27$.

Theorem: General solution - nonhomogeneous equations

Let y_p be any particular solution of the nonhomogeneous linear nth-order DE (4) on an interval I, and let $y_1, y_2, ..., y_n$ be a fundamental set of solutions of the associated homogeneous DE (3) on I. Then the **general solution** of the equation on I is

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x) + y_p \tag{7}$$

where the $c_i, i = 1, 2, ..., n$ are arbitrary constants.

Complementary function

The general solution of a homogeneous linear equation consists of the sum of two functions

$$y = c_1y_1(x) + c_2y_2(x) + ... + c_ny_n(x) + y_p(x) = y_c(x) + y_p(x)$$

The linear combination $y = c_1y_1(x)+c_2y_2(x)+...+c_ny_n(x)$ which is the general solution of the homogeneous DE (3), is called the **complementary solution** for equation (4).

Thus to solve the nonhomogeneous linear DE, we first solve the associated homogeneous equation and then find any particular solution of the nonhomogeneous equation. The general solution is then

y = complementary function + any particular solution.

Another superposition principle

Theorem: Superposition principle - nonhomogeneous equations

Let $y_{p_1}, y_{p_2}, ..., y_{p_k}$ be k particular solutions of the nonhomogeneous linear nth-order DE (4) on an interval I corresponding, in turn, to k distinct functions $g_1, g_2, ..., g_k$. That is, suppose y_{p_i} denotes a particular solution of the corresponding DE

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ . \ . \ . \ + a_1(x)y' + a_0(x)y = g_i(x) \tag{8}$$

where $i = 1, 2, ..., k$. Then

$$y_p = y_{p_1}(x) + y_{p_2}(x) + ... + y_{p_k}(x) \tag{9}$$

is a particular solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ . \ . \ . \ + a_1(x)y' + a_0(x)y = g_1(x) + g_2(x) + ... + g_k(x) \tag{10}$$

For proof for the case $k = 2$ see D.G. Zill et al., Advanced Engineering Mathematics, 4th Edition, p. 104.

Example:

Verify that

$$\begin{aligned} y_{p_1} &= -4x^2 && \text{is a particular solution of} \quad y'' - 3y' + 4y = -16x^2 + 24x - 8 \\ y_{p_2} &= e^{2x} && \text{is a particular solution of} \quad y'' - 3y' + 4y = 2e^{2x} \\ y_{p_3} &= xe^{x} && \text{is a particular solution of} \quad y'' - 3y' + 4y = 2xe^{x} - e^{x} \end{aligned}$$

and that $y = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^{x}$ is a solution of

$$y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 2xe^{x} - e^{x}$$

Remarks:

A dynamical system whose mathematical model is a linear nth-order DE

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ \ldots \ + a_1(t)y' + a_0(t)y = g(t)$$

is said to be a **linear system**. The set of n time dependent functions $y(t), y'(t)$, ... , $y^{(n-1)}(t)$ are the **state variables** of the system. Their values at some time t give the **state of the system**. The function g is called the **input function**, **forcing function**, or **excitation function**. A solution $y(t)$ of the DE is said to be the **output** or **response of the system**. The output or response $y(t)$ is uniquely determined by the input and the state of the system prescribed at a time t_0; that is, by the initial conditions $y(t_0), y'(t_0), ..., y^{(n-1)}(t_0)$.

Reduction of order

Suppose $y(x)$ denotes a known solution of a homogeneous linear second-order equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{11}$$

we seek the second solution $y_2(x)$ so that y_1 and y_2 are linearly independent on some interval I. That is we are looking for y_2 s. t. $y_2/y_1 = u(x)$, or $y_2(x) = u(x)y_1(x)$.

The idea is to find $u(x)$ by substituting $y_2(x) = u(x)y_1(x)$ into the DE. This method is called **reduction of order** since we must solve a first-order equation to find u.

Example:

Given $y_1 = e^x$ is a solution of $y'' - y = 0$ on $(-\infty, \infty)$, use the reductions of order to find a second solution y_2.

General case

We put the equation (11) into the standard form by dividing by $a_2(x)$:

$$y'' + P(x)y' + Q(x)y = 0 \tag{12}$$

where $P(x)$ and $Q(x)$ are continuous on some interval I. Assume that $y_1(x)$ is a known solution of (12) on I and that $y_1(x) \neq 0$ for every $x \in I$. We define $y = u(x)y_1(x)$

$$y' = uy_1' + y_1u', \qquad y'' = uy_1'' + 2y_1'u' + y_1u''$$
$$y'' + Py' + Qy = u\left[y_1'' + Py_1' + Qy_1\right] + y_1u'' + \left(2y_1' + Py_1\right)u' = 0$$

where the term in the square bracket equals to zero.

This implies

$$y_1 u'' + \left(2y_1' + Py_1\right) u' = 0 \quad \text{or} \quad y_1 w' + \left(2y_1' + Py_1\right) w = 0$$

where we used $w = u'$. The last equation can be solved by separating variables and integrating

$$\frac{dw}{w} + 2\frac{y_1'}{y_1}dx + Pdx = 0$$

$$\ln\left|wy_1^2\right| = -\int Pdx + c$$

or

$$wy_1^2 = c_1 e^{-\int Pdx}$$

Solving the last equation for w, and using $w = u'$ and integrating again gives

$$u = c_1 \int \frac{e^{-\int P dx}}{y_1^2} dx + c_2$$

By choosing $c_1 = 1$ and $c_2 = 0$ and by using $y = u(x)y_1(x)$ we find the second solution of the equation (12):

$$y_2 = y_1(x) \int \frac{e^{-\int P dx}}{y_1(x)^2} dx \tag{13}$$

Example:

$y_1 = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$. Find the general solution on $(0, \infty)$.

From the standard form of the equation

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

we find using the formula above

$$y_2 = x^2 \int \frac{e^{3\int dx/x}}{x^4} dx = x^2 \int \frac{dx}{x} = x^2 \ln x$$

The general solution on $(0, \infty)$ is given by

$$y = c_1y_1 + c_2y_2 = c_1x^2 + c_2x^2 \ln x$$

Homogeneous linear equations with constant coefficients

All solutions of the homogeneous linear higher-order DE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

where the coefficients $a_i, i = 0, 1..., n$ are real constants and $a_n \neq 0$ are either exponential functions or are constructed out of exponential functions.

Recall: the solution of the linear first-order DE $y' + ay = 0$, where a is a constant, has an exponential solution $y = c_1 e^{-ax}$ on $(-\infty, \infty)$.

Auxiliary equation

We focus on the second-order equation

$$ay'' + by' + cy = 0 \tag{14}$$

If we try a solution $y = e^{mx}$, the equation above becomes

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0 \quad \text{or} \quad e^{mx}(am^2 + bm + c) = 0$$

Since e^{mx} is never zero for real values of x, the exponential function can satisfy the DE (14) only if m is a root of the quadratic equation

$$am^2 + bm + c = 0$$

which is called the **auxiliary equation**.

Since the roots of the auxiliary equation are

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$
$$m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

there will be three forms of general solution of (14):

- m_1 and m_2 are real and distinct $(b^2 - 4ac > 0)$,
- m_1 and m_2 are real and equal $(b^2 - 4ac = 0)$, and
- m_1 and m_2 are conjugate complex numbers $(b^2 - 4ac < 0)$

Distinct real roots

We have two solutions $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ which are linearly independent on $(-\infty, \infty)$ and thus form a fundamental set of solutions.

The general solution is on this interval

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Repeated roots

When $m_1 = m_2$ we get only one exponential solution $y_1 = e^{m_1 x}$ where $m_1 = -b/2a$ ($b^2 - 4ac = 0$ in the expression for the roots of the quadratic equation).

The second solution can be found by reduction of order:

$$y_2 = e^{m_1 x} \int \frac{e^{2m_1 x}}{e^{2m_1 x}} dx = e^{m_1 x} \int dx = x e^{m_1 x}$$

where we used $-P(x) = -b/a = 2m_1$.

$ay'' + by' + cy = 0$

The generals solution is

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

Conjugate complex roots

We can write $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ where α and $\beta > 0$ are real and $i^2 = -1$. Formally this case is similar to the case I:

$$y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$$

Since this is a solution for any choice of the constants C_1 and C_2, the choices $C_1 = C_2 = 1$ and $C_1 = 1$ and $C_2 = -1$ give two solutions

$$\begin{aligned} y_1 &= e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x} = e^{\alpha x}\left(e^{i\beta x} + e^{-i\beta x}\right) = 2e^{\alpha x}\cos\beta x \\ y_2 &= e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x} = e^{\alpha x}\left(e^{i\beta x} - e^{-i\beta x}\right) = 2ie^{\alpha x}\sin\beta x \end{aligned}$$

where we used the Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

The last two results show that $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are real solutions of (14) and form the fundamental set on $(-\infty, \infty)$. The general solution is

$$y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x).$$

Example 1: Solve the following DEs:

$$\begin{aligned} 2y'' - 5y' - 3y &= 0 \\ y'' - 10y' + 25y &= 0 \\ y'' + 4y' + 7y &= 0 \end{aligned}$$

Example 2: Solve the following IVP:

$$4y'' + 4y' + 17y = 0, \qquad y(0) = -1, \;\; y'(0) = 2$$

Example 3:

$$y'' + k^2 y = 0 \qquad y'' - k^2 y = 0$$

where k is real.

Undetermined coefficients

To solve a nonhomogeneous linear DE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = g(x)$$

we must

- find the complementary function y_c; and
- find *any* particular solution y_p of the nonhomogeneous equation.

The general solution on an interval I is $y = y_c + y_p$ where y_c is the solution of the associated homogeneous DE:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

Method of undetermined coefficients

To obtain a particular solution y_p we will make an educated guess about the form of y_p motivated by the kind of function that makes up the input function $g(x)$.

The general method is limited to nonhomogeneous linear DE where

- the coefficients, $a_i, i = 0, 1, ..., n$ are constants, and
- where $g(x)$ is a constant, a polynomial function, an exponential function $e^{\alpha x}$, sine or cosine functions $\sin\beta x$ or $\cos\beta x$, or finite sums and products of these functions.

The method of undetermined coefficients is not applicable to equations of the form (15) if

$$g(x) = \ln x, \quad g(x) = \frac{1}{x}, \quad g(x) = \tan x, \quad g(x) = \sin^{-1} x$$

Example 1: Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6$.

Solution:

Step 1: Solve associated homogeneous equation $y'' + 4y' - 2y = 0$.
We find the roots of the auxiliary equatin $m^2 + 4m - 2 = 0$ are $m_1 = -2 - \sqrt{6}$ and $m_2 = -2 + \sqrt{6}$. The complementary function is thus

$$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}$$

$$y'' + 4y' - 2y = 2x^2 - 3x + 6$$

Step 2: Since $g(x)$ is quadratic polynomial, let us assume a particular solution in the form

$$y_p = Ax^2 + Bx + C$$

We wish to determine the coefficients A, B, and C for which y_p is a solution of the equation above:

$$y_p'' + 4y_p' - 2y_p = 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6$$

The coefficients of like powers of x must be equal, that is

$$-2A = 2, \quad 8A - 2B = -3, \quad 2A + 4B - 2C = 6$$

This leads to $A = -1$, $B = -5/2$, and $C = -9$, so this particular solution is

$$y_p = -x^2 - \frac{5}{2}x - 9. \tag{15}$$

Step 3: The general solution is then

$$y = y_c + y_p = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9. \tag{16}$$

Example 2: Particular solution using undetermined coefficients
Find a particular solution of $y'' - y' + y = 2\sin 3x$.

Example 3: Forming y_p by superposition
Find a particular solution of $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$.

A glitch in the method:

Example 4: Find a particular solution of $y'' - 5y' + 4y = 8e^x$.

Differentiation of e^x produces no new function, so proceeding with the particular solution assumed in the form of $y_p = Ae^x$ leads to a contradiction $0 = 8e^x$.

In fact our y_p is already contained in $y_c = c_1e^x + c_2e^{4x}$. Let us see whether we can find a particular solution of the form

$$y_p = Axe^x$$

Substituting this solution into the DE and simplifying gives

$$y_p'' - 5y_p' + 4y_p = -3Ae^x = 8e^x \quad \text{so} \quad y_p = -\frac{8}{3}xe^x$$

We distinguish two cases:

Case I: No function in the assumed particular solution is a solution of the associated homogeneous differential equation.

Case II: A function in the assumed particular solution is also a solution of the associated homogeneous differential equation.

Trial particular solutions

	$g(x)$	Form of y_p
1.	1 (any constant)	A
2.	$5x + 7$	$Ax + B$
3.	$3x^2 - 2$	$Ax^2 + Bx + C$
4.	$x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5.	$\sin 4x$	$A\cos 4x + B\sin 4x$
6.	$\cos 4x$	$A\cos 4x + B\sin 4x$
7.	e^{5x}	Ae^{5x}
8.	$(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9.	x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10.	$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11.	$5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
12.	$xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$

Example 5: Forms of particular solution - Case I

Determine the form of a particular solution of

$$\begin{aligned} y'' - 8y' + 25y &= 5x^3 e^{-x} - e^{-x} \\ y'' + 4y &= x \cos x \end{aligned}$$

If $g(x)$ consists of a sum of m terms of the kind listed in the table above, the assumption for a particular solution y_p consists of the sum of the trial forms $y_{p_1}, y_{p_2}, ..., y_{p_n}$ corresponding to these terms:

$$y_p = y_{p_1} + y_{p_2} + ... + y_{p_n}$$

The form rule for Case I: The form of y_p is a linear combination of all linearly independent functions that are generated by repeated differentiations of $g(x)$.

Example 6: Forming y_p by superposition - Case I

Determine the form of a particular solution of

$$y'' - 9y' + 14y = 3x^2 - 5\sin 2x + 7xe^{6x}$$

Solution:

$$\begin{aligned}
3x^2 &\Rightarrow y_{p_1} = Ax^2 + Bx + C \\
-5\sin 2x &\Rightarrow y_{p_2} = E\cos 2x + F\sin 2x \\
7xe^{6x} &\Rightarrow y_{p_3} = (Gx + H)e^{6x}
\end{aligned}$$

$$y = y_{p_1} + y_{p_2} + y_{p_3} = Ax^2 + Bx + C + E\cos 2x + F\sin 2x + (Gx + H)e^{6x}$$

No term in this solution duplicates a term in $y_c = c_1e^{2x} + c_2e^{7x}$.

Example 7: Particular solution - Case II

Find a particular solution of

$$y'' - 2y' + y = e^x$$

The complementary function is $y_c = c_1 e^x + c2xe^x$. Therefore we can not assume the particular solution in the form $y_p = Ae^x$ or $y_p = Axe^x$ since these would duplicate the terms in y_c. We try

$$y_p = Ax^2 e^x$$

Substituting this into the DE gives $2Ae^x = e^x$ and so $A = \frac{1}{2}$. The particular solution is $y_p = \frac{1}{2}x^2 e^x$.

Suppose again that $g(x)$ consists of m terms given by the table above, and that a particular solution y_p consists of the sum:

$$y_p = y_{p_1} + y_{p_2} + ... + y_{p_n}$$

where $y_{p_i}, i = 1, 2, ..., m$ are the corresponding trial solution forms.

Multiplication rule for Case II: If any y_{p_i} contains terms that duplicate terms in y_c then that y_{p_i} must be multiplied by x^n, where n is the smallest positive integer that eliminates that duplication.

Example 8: An IVP

$$y'' + y = 4x + 10\sin x, \qquad y(\pi) = 0, \;\; y'(\pi) = 2 \tag{17}$$

The solution of the associated homogeneous equation $y'' + y = 0$ is $y_c = c_1 \cos x + c_2 \sin x$. To avoid duplication we use

$$y_p = Ax + B + Cx\cos x + Ex\sin x$$

The final solution of the IVP:

$$y = 9\pi\cos x + 7\sin x + 4x - 5x\cos x$$

Example 9: Using the multiplication rule, solve

$$y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$$

The solution of the associated homogeneous equation is $y_c = c_1 e^{3x} + c_2 x e^{3x}$, so we choose the operative form of the particular solution to be

$$y_p = Ax^2 + Bx + C + Ex^2 e^{3x}$$

Substituting into the differential equation and collecting like terms gives $A = \frac{2}{3}$, $B = \frac{8}{9}$, $C = \frac{2}{3}$, and $E = -6$. The general solution is then

$$y = y_c + y_p = c_1 e^{3x} + c_2 x e^{3x} + \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3} - 6x^2 e^{3x}$$

The method of variation of parameters

Advantage: the method always yields a particular solution y_p, provided the associated homogeneous equation can be solved. Also it is not limited to certain types of $g(x)$.

First we put a linear second-order DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \tag{18}$$

into the standard form by dividing by $a_2(x)$

$$y'' + P(x)y' + Q(x)y = f(x) \tag{19}$$

We seek a solution of the form

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where y_1 and y_2 form a fundamental set of solutions on I of the associated homogeneous form of (18). Using the product rule to differentiate y_p twice gives

$$\begin{aligned} y_p' &= u_1y_1' + y_1u_1' + u_2y_2' + y_2u_2' \\ y_p'' &= u_1y_1'' + y_1'u_1' + y_1u_1'' + u_1'y_1' + u_2y_2'' + y_2'u_2' + y_2u_2'' + u_2'y_2' \end{aligned}$$

Substituting these into the standard form (19) yields

$$\begin{aligned} y_p'' + P(x)y_p' + Q(x)y_p &= u_1\left[y_1'' + Py_1' + Qy_1\right] + u_2\left[y_2'' + Py_2' + Qy_2\right] \\ &\quad + y_1u_1'' + u_1'y_1' + y_2u_2'' + u_2'y_2' + P\left[y_1u_1' + y_2u_2'\right] + y_1'u_1' + y_2'u_2' \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[y_1u_1'\right] + \frac{\mathrm{d}}{\mathrm{d}x}\left[y_2u_2'\right] + P\left[y_1u_1' + y_2u_2'\right] + y_1'u_1' + y_2'u_2' \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[y_1u_1' + y_2u_2'\right] + P\left[y_1u_1' + y_2u_2'\right] + y_1'u_1' + y_2'u_2' = f(x) \end{aligned} \tag{20}$$

We need two equations for two unknown functions u_1 and u_2. Assuming that these functions satisfy $y_1u_1' + y_2u_2' = 0$, the equation above reduces to $y_1'u_1' + y_2'u_2' = f(x)$. By Cramer's rule, the solution of the system

$$\begin{aligned} y_1u_1' + y_2u_2' &= 0 \\ y_1'u_1' + y_2'u_2' &= f(x) \end{aligned}$$

can be expressed in terms of determinants:

$$u_1' = \frac{W_1}{W} = -\frac{y_2f(x)}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W} = \frac{y_1f(x)}{W} \tag{21}$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}.$$

The functions u_1 and u_2 are found by integrating the result in (21). The determinant W is the Wronskian of y_1 and y_2 whose linear independence ensures that $W \neq 0$.

Example: General solution using variation of parameters

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

From the auxiliary equation $m^2 - 4m + 4 = (m-2)^2 = 0$ we have $y_c = c_1 e^{2x} + c_2 x e^{2x}$. We identify $y_1 = e^{2x}$ and $y_2 = xe^{2x}$ and evaluate the Wronskian

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}$$

The DE above is already in the standard form, so $f(x) = (x+1)e^{2x}$, W_1 and W_2 are then

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -(x+1)xe^{4x}, \quad W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}$$

and so

$$u_1' = -\frac{(x+1)xe^{4x}}{e^{4x}} = -x^2 - x, \qquad u_2' = \frac{(x+1)e^{4x}}{e^{4x}} = x + 1$$

It follows that $u_1 = -\frac{1}{3}x^3 - \frac{1}{2}x^2$ and $u_2 = \frac{1}{2}x^2 + x$, and hence

$$y_p = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x} = \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

The general solution is then

$$y = y_c + y_p = c_1e^{2x} + c_2xe^{2x} + \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

Linear models: initial value problem

(1) **Spring-mass problem: free undamped motion**

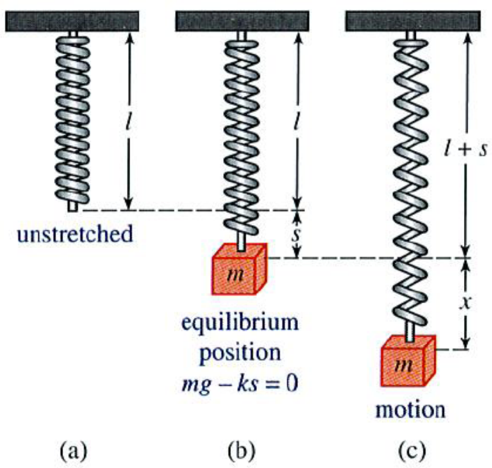
Newton's law

$$F = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}$$

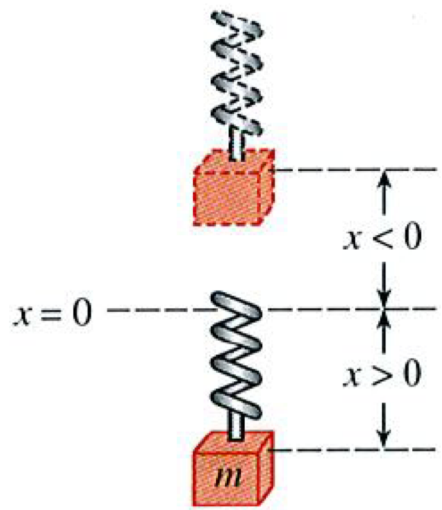
Hook's law

$$F = -kx$$

By putting these two laws together we get the desired ODE



$$m\frac{d^2x}{dt^2} + kx = 0$$



If we divide the equation by mass m and introduce the angular frequency $\omega = \sqrt{k/m}$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x = 0$$

we have a homogeneous linear second-order which describes **simple harmonic motion** or **free undamped motion**.

The initial conditions associated with the DE above are the amount of initial displacement $x(0) = x_0$, and the initial velocity of the mass $x'(0) = x_1$.

To solve the equation, we note that the auxiliary equation $m^2 + \omega^2 = 0$ has two complex roots $m_1 = i\omega$ and $m_2 = -i\omega$, so the general solution is to be

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

We determine c_1 and c_2 from the initial condition and obtain the **equation of motion**.

Example: The equation of motion

$$x(t) = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t$$

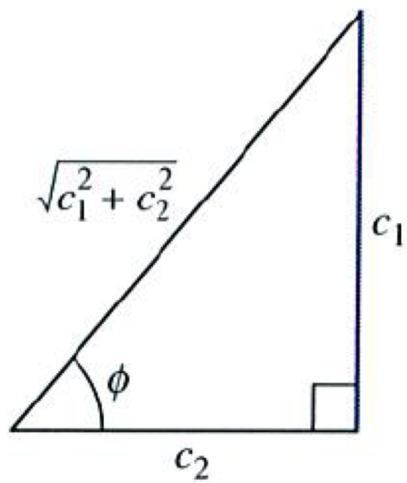
Angular frequency: $\omega = 8$
Period: $T = 2\pi/\omega = 2\pi/8 = \pi/4$
Frequency: $f = 1/T = 4/\pi$

Alternative form of $x(t)$:

$$x(t) = A sin(\omega t + \phi)$$

where $A = \sqrt{c_1 + c_2}$ is the **amplitude** of free vibrations, and ϕ is the **phase angle** defined by

$$\sin\phi = \frac{c_1}{A}, \qquad \cos\phi = \frac{c_2}{A}, \qquad \Rightarrow \qquad \tan\phi = \frac{c_1}{c_2}$$

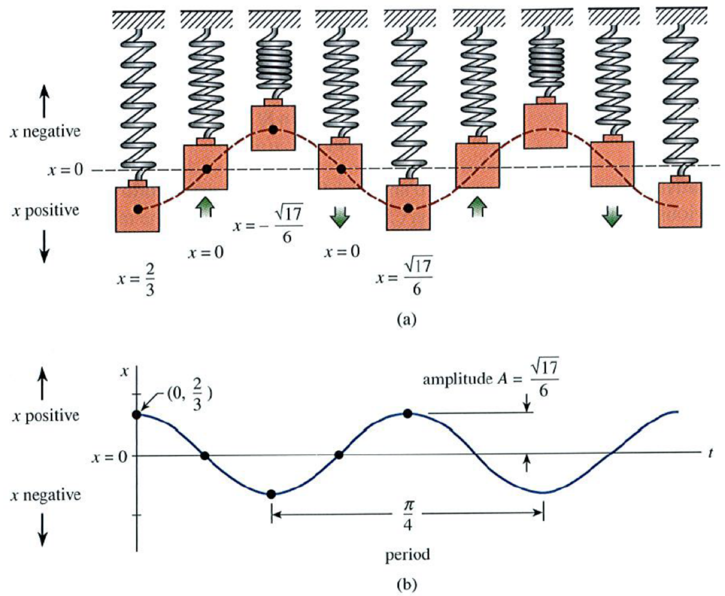


To see the relation between the original solution and its alternative form, we use trigonometry

$$
\begin{aligned}
x(t) &= c_1 \cos \omega t + c_2 \sin \omega t \\
&= A \sin \phi \cos \omega t + A \cos \phi \sin \omega t \\
&= A \sin(\omega t + \phi)
\end{aligned}
$$

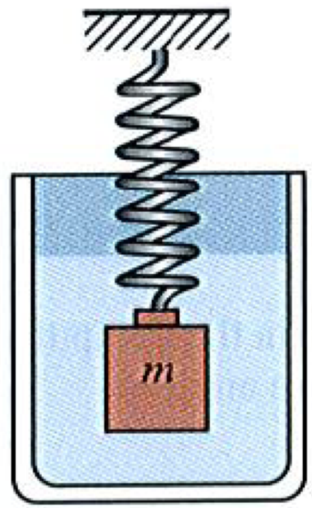
In our specific example, we get

$$
\begin{aligned}
x(t) &= \frac{2}{3} \cos 8t - \frac{1}{6} \sin 8t \\
&= \frac{\sqrt{17}}{6} \sin(8t + 1.816)
\end{aligned}
$$



(2) **Spring-mass problem: free damped motion**

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}$$



By dividing by the mass m we get the DE of **free damped motion**:

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega^2 x = 0$$

The corresponding auxiliary equation $m^2 + 2\gamma m + \omega^2 = 0$ has the roots

$$m_1 = -\gamma + \sqrt{\gamma^2 - \omega^2} \quad \text{and} \quad m_2 = -\gamma - \sqrt{\gamma^2 - \omega^2}$$

Each solution will contain the **damping factor** $e^{-\gamma t}$, $\gamma > 0$ and thus the displacements of the mass become negligible over time.

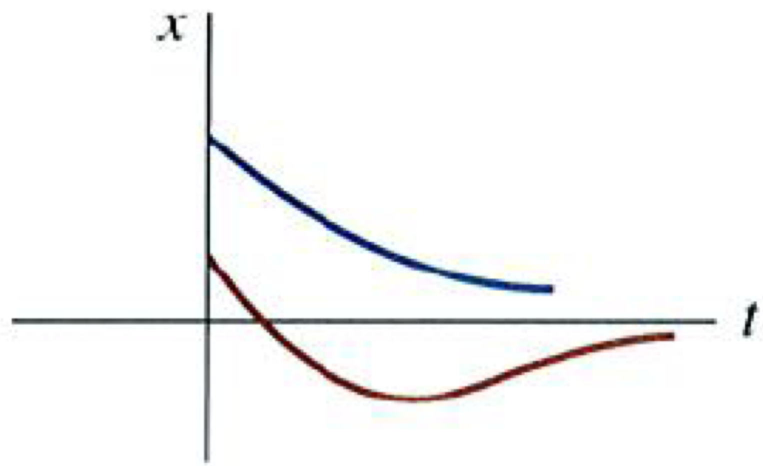
Depending on the algebraic sign of $\gamma^2 - \omega^2$, we distinguish three cases:

- **Case I:** $\gamma^2 - \omega^2 > 0$

  In this case the system is **overdamped**, as the damping coefficient β is large compared to the spring constant k.

  The corresponding solution $x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$ is

$$x(t) = e^{-\gamma t}\left(c_1 e^{\sqrt{\gamma^2-\omega^2}t} + c_2 e^{-\sqrt{\gamma^2-\omega^2}t}\right)$$

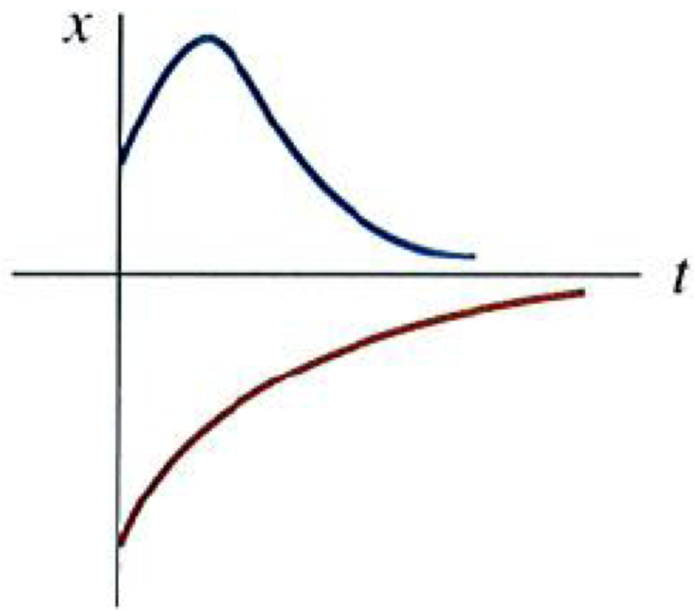


- **Case II:** $\gamma^2 - \omega^2 = 0$

  In this case the system is **critically damped**, because a slight decrease of the damping would result in oscillatory motion.

  The general solution $x(t) = c_1 e^{m_1 t} + c_2 t e^{m_2 t}$ is

$$x(t) = e^{-\gamma t}(c_1 + c_2 t)$$



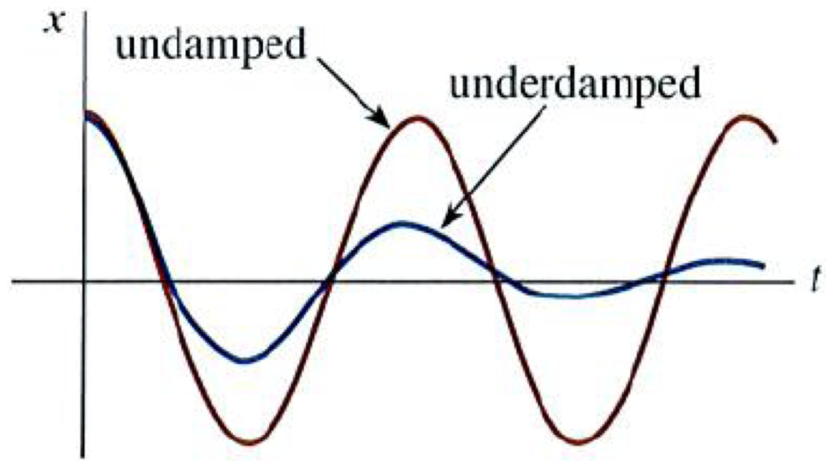
- **Case III:** $\gamma^2 - \omega^2 < 0$

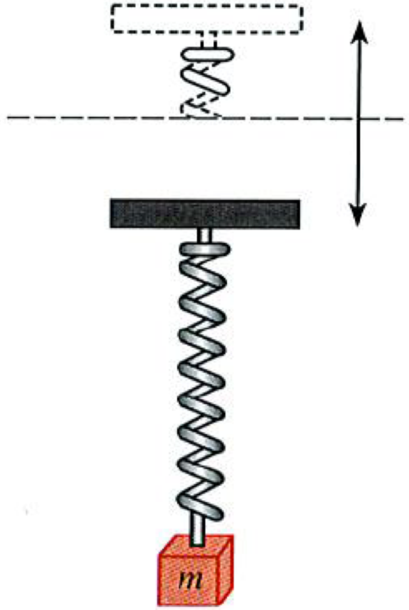
  In this case the system is **underdamped**, as the damping coefficient is small compared to the spring constant. The roots of the auxiliary equation are now complex:

  $$m_1 = -\gamma + i\sqrt{\omega^2 - \gamma^2} \quad \text{and} \quad m_2 = -\gamma - i\sqrt{\omega^2 - \gamma^2}$$

  and thus the general solution is

  $$x(t) = e^{-\gamma t}\left(c_1 \cos\sqrt{\omega^2 - \gamma^2}\, t + c_2 \sin\sqrt{\omega^2 - \gamma^2}\, t\right)$$





(3) **Spring-mass problem: driven motion**

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \beta\frac{\mathrm{d}x}{\mathrm{d}t} + kx \;=\; f(t)$$

By dividing by the mass m we get the DE of **driven motion**:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 x \;=\; F(t)$$

which is a nonhomogeneous differential equation whose solution can be obtained either using

- the *method of undetermined coefficients*, or
- the *method of variation of parameters*.

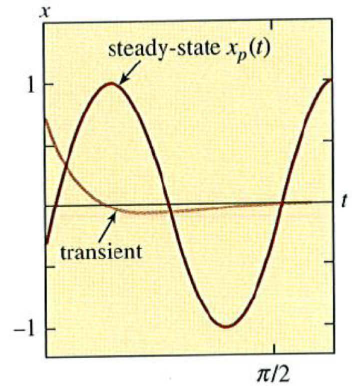
Example: Transient/Steady-state solutions

The solution of the IVP

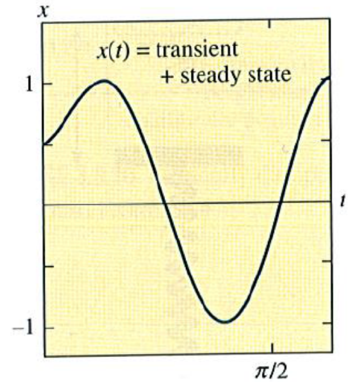
$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 10x = 25\cos 4t, \qquad x(0) = \frac{1}{2},\ \ x'(0) = 0$$

is given by

$$x(t) = x_c + x_p = e^{-3t}\left(\frac{38}{51}\cos t - \frac{86}{51}\sin t\right) - \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t$$



(a)



(b)

where the first term represents the **transient** solution and the remaining two terms are the **steady state** solution of the IVP.

Example: Undamped forced motion

Consider the IVP

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega_0^2 x = F_0 \sin\omega t, \qquad x(0) = 0, \quad x'(0) = 0$$

the complementary solution is $x_c(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$. We assume the particular solution in the form $x_p = A\cos\omega t + B\sin\omega t$, so that

$$x_p'' + \omega_0^2 x_p = A\left(\omega_0^2 - \omega^2\right)\cos\omega t + B\left(\omega_0^2 - \omega^2\right)\sin\omega t = F_0 \sin\omega t$$

Equating coefficients gives $A = 0$ and $B = F_0/\left(\omega_0^2 - \omega^2\right)$, and thus the general solution is

$$x(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + \frac{F_0}{\left(\omega_0^2 - \omega^2\right)} \sin\omega t$$

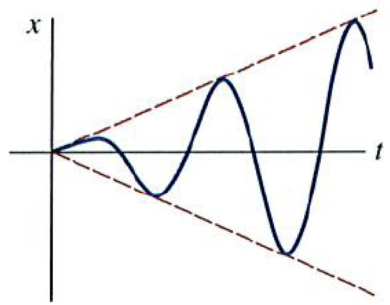
The initial conditions yield $c_1 = 0$ and $c_2 = -\omega F_0/\omega_0\left(\omega_0^2 - \omega^2\right)$, so the solution of the IVP is

$$x(t) = \frac{F_0}{\omega_0\left(\omega_0^2 - \omega^2\right)}(-\omega \sin \omega_0 t + \omega_0 \sin \omega t)$$

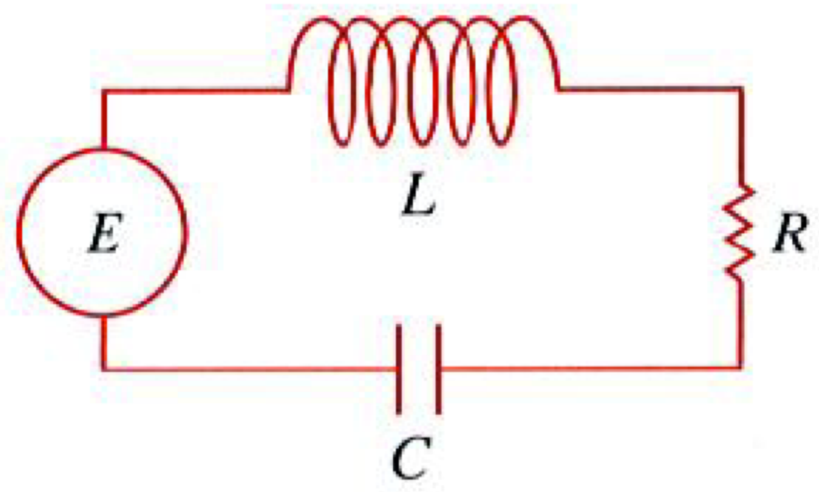
Though the equation is not defined for $\omega = \omega_0$, the limit $\omega \to \omega_0$ can be calculated using he L'Hospital rule giving

$$x(t) = \lim_{\omega \to \omega_0} F_0 \frac{-\omega \sin \omega_0 t + \omega_0 \sin \omega t}{\omega_0\left(\omega_0^2 - \omega^2\right)} = \frac{F_0}{2\omega_0^2} \sin \omega_0 t - \frac{F_0}{2\omega_0}\, t \, \cos \omega_0 t$$

As time increases, so does the response of the system to the driving and the displacements become large. This is the phenomenon of **pure resonance**.



LRC-series electric circuit



$i(t)$ - the current in a circuit at time t

$q(t)$ - the charge on the capacitor at time t

L - inductance

C - capacitance

R - resistance

According to **Kirchhoff's second law**, the impressed voltage $E(t)$ must equal to the sum of the voltage drops in the loop.

$$V_L + V_C + V_R = E(t)$$

Inductor

$$V_L = L\frac{\mathrm{d}i}{\mathrm{d}t} = L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2}$$

Capacitor

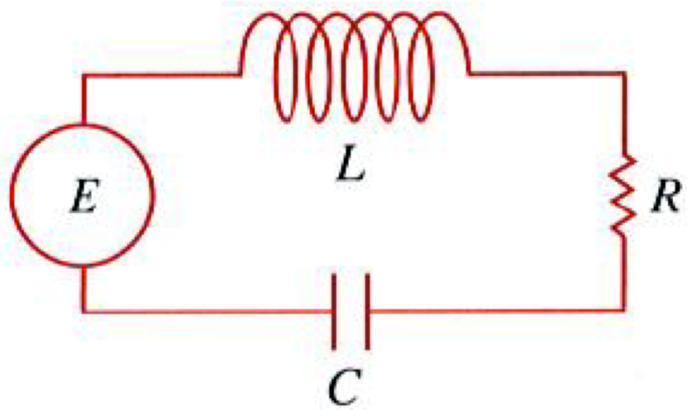
$$V_C = \frac{q}{C}$$

Resistor

$$V_R = Ri = R\frac{\mathrm{d}q}{\mathrm{d}t}$$

LRC circuit

$$L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q = E(t)$$



Example: LRC circuit

Find the steady-state solution q_p and the **steady-state current** in an LRC-series circuit when the driving voltage is $E(t) = E_0 \sin \omega t$.

The steady-state solution q_p is a particular solution of the differential equation

$$L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q = E(t)$$

Using the method of undetermined coefficients, we assume the particular solution of the form $q_p(t) = A \sin \omega t + B \cos \omega t$. Substituting this into the DE, simplifying and equating coefficients gives

$$A = \frac{E_0\left(L\omega - \frac{1}{C\omega}\right)}{-\omega\left(L^2\omega^2 - \frac{2L}{C} + \frac{1}{C^2\omega^2} + R^2\right)}, \quad B = \frac{E_0 R}{-\omega\left(L^2\omega^2 - \frac{2L}{C} + \frac{1}{C^2\omega^2} + R^2\right)}$$

It is convenient to express this using the **reactance** $X = L\omega - 1/(C\omega)$ and the **impedance** $Z = \sqrt{X^2 + R^2}$ (both measured in ohms). We get

$$A = \frac{E_0 X}{-\omega Z^2}, \qquad B = \frac{E_0 R}{-\omega Z^2}$$

so the steady state charge is

$$q_p(t) = -\frac{E_0 X}{\omega Z^2} \sin \omega t - \frac{E_0 R}{\omega Z^2} \cos \omega t$$

and the steady-state current $i_p(t) = q_p'(t)$

$$i_p(t) = \frac{E_0}{Z}\left(\frac{R}{Z} \sin \omega t - \frac{X}{Z} \cos \omega t\right)$$