Introduction to differential equations: overview

- Definition of differential equations and their classification
- Solutions of differential equations
- Initial value problems
- Existence and uniqueness
- Mathematical models and examples
- Methods of solution of first-order differential equations

Definition: Differential Equation

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation (DE)**:

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \dots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

<u>Examples:</u>

$(i)\ \dfrac{\mathrm{d}^4 y}{\mathrm{d}x^4} + y^2 = 0 \qquad (ii)\ y'' - 2y' + y = 0 \qquad (iii)\ \ddot{s} = -32 \qquad (iv)\ \dfrac{\partial^2 u}{\partial x^2} = -2\dfrac{\partial u}{\partial t}$

Classification of differential equations

(a) Classification by Type:

Ordinary differential equations - ODE

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0$$

Partial differential equations - PDE

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$$

(b) Classification by Order:

The **order** of the differential equation is the order of the highest derivative in the equation.

Example:

nth-order ODE:

$$F\left(x, y, y', ..., y^{(n)}\right) = 0 \tag{1}$$

Normal form of (1)

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = f\left(x, y, y' ..., y^{(n-1)}\right)$$

(c) Classification as Linear or Non-linear:

An nth-order ODE (1) is said to be **linear** if it can be written in this form

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + ... + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

<u>Examples:</u>

Linear: $(y - x)dx + 4xdy = 0$ $\qquad y'' - 2y' + y = 0$ $\qquad \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 3x\frac{\mathrm{d}y}{\mathrm{d}x} - 5y = e^x$

Nonlinear: $\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + y^2 = 0$ $\qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \sin(y) = 0$ $\qquad (1 - y)y' + 2y = e^x$

Solution of an ODE:

Any function ϕ defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an n-th-order ordinary differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

In other words:
a solution of an nth-order ODE is a function ϕ that possesses at least n derivatives and

$$F\left(x, \phi(x), \phi'(x), ..., \phi^{(n)}(x)\right) = 0 \tag{2}$$

for all $x \in I$. Alternatively we can denote the solution as $y(x)$.

Interval of definition:

A *solution* of an ODE has to be considered simultaneously with the *interval* I which we call

the interval of definition
the interval of existence,
the interval of validity, or
the domain of the solution.

It can be an open interval (a, b), a closed interval $[a, b]$, an infinite interval (a, ∞) and so on.

Example:

Verify that the function $y = xe^x$ is a solution of the differential equation $y'' - 2y' + y = 0$ on the interval $(-\infty, \infty)$:

From the derivatives

$$\begin{aligned} y' &= xe^x + e^x \\ y'' &= xe^x + 2e^x \end{aligned}$$

we see

$$\begin{aligned} &l.h.s. : \qquad y'' - 2y' + y = (xe^x + 2e^x) - 2\,(xe^x + e^x) + xe^x = 0 \\ &r.h.s. : \qquad 0 \end{aligned}$$

that each side of the equation is the same for every real number x.

A solution that is identically zero on an interval I, i.e. $y = 0, \forall x \in I$, is said to be **trivial**.

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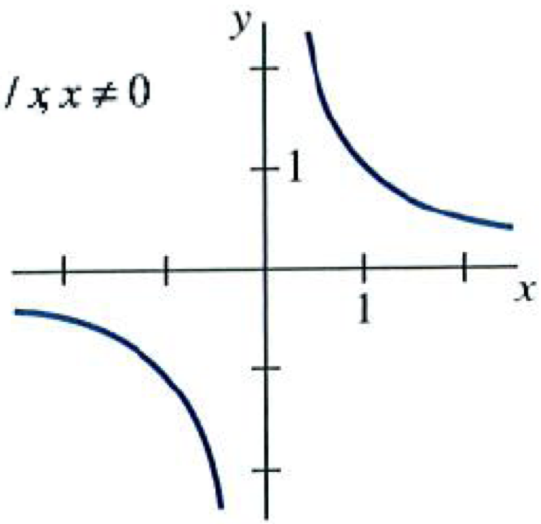
Solution curve:

is the graph of a solution ϕ of an ODE.

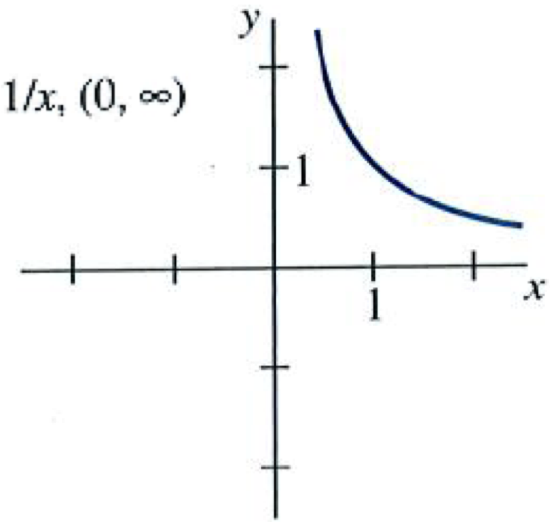
The graph of the solution ϕ is NOT the same as the graph of the functions ϕ as the domain of the function ϕ does not need to be the same as the interval I of definition (domain) of the solution ϕ.

<u>Example:</u>

(a) Function $y = 1/x$, $x \neq 0$



(b) Solution $y = 1/x$, $(0, \infty)$



Explicit solutions:

a solution in which the dependent variable is expressed solely in terms of the independent variable and constants.

Example:

$$y = \phi(x) = e^{0.1\ x^2}$$

is an explicit solution of the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0.2xy$$

Implicit solutions:

A relation $G(x, y) = 0$ is said to be an **implicit solution** of an ODE on an interval I provided there exists at least one function ϕ that satisfies the relation as well as the differential equation on I.

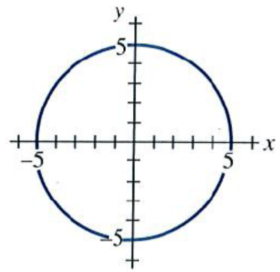
<u>Example:</u>

$$x^2 + y^2 = 25$$

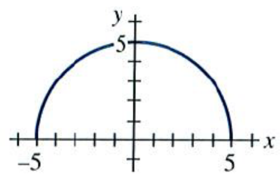
is an implicit solution of the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$$

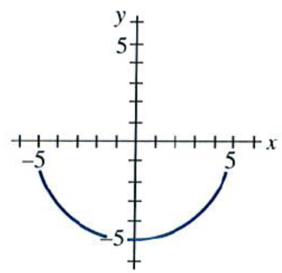
on the interval $(-5, 5)$.
Notice that also $x^2 + y^2 - c = 0$ satisfies the ODE above.



(a) Implicit solution
$x^2 + y^2 = 25$



(b) Explicit solution
$y_1 = \sqrt{25 - x^2}, \ -5 < x < 5$



(c) Explicit solution
$y_2 = -\sqrt{25 - x^2}, \ -5 < x < 5$

Families of solutions:

A solution ϕ of a first-order ODE $F(x, y, y') = 0$ can be referred to as an **integral** of the equation, and its graph is called an **integral curve**.

A solution containing an arbitrary constant (an integration constant) c represents a set

$$G(x, y, c) = 0$$

called a **one-parameter family of solutions**.

When solving an nth-order ODE $F(x, y, y', ..., y^{(n)}) = 0$, we seek an n**-parameter family of solutions** $G(x, y, c_1, c_2, ..., c_n) = 0$.

A single ODE can possess an infinite number of solutions!

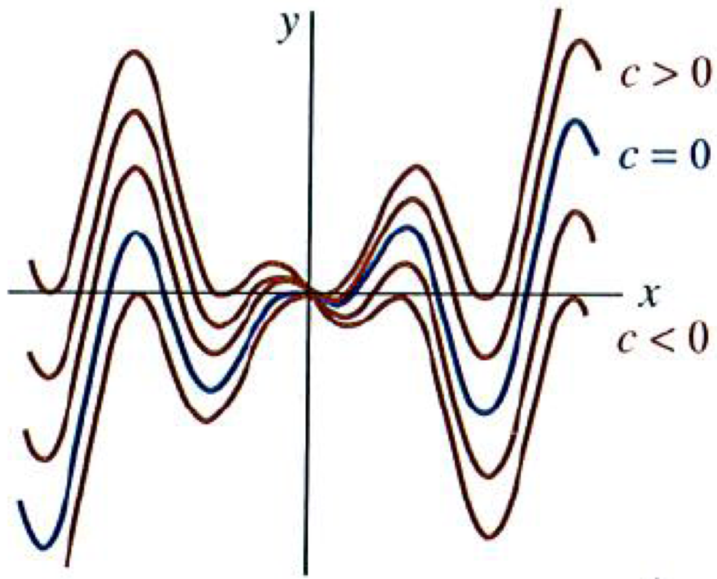
A particular solution:

is a solution of an ODE that is free of arbitrary parameters.

Example:

$y = cx - x\cos x$ is an explicit solution of $xy' - y = x^2 \sin x$ on $(-\infty, \infty)$.

The solution $y = -x\cos x$ is a particular solution corresponding to $c = 0$.



A singular solution:

a solution that can not be obtained by specializing any of the parameters in the family of solutions.

Example:

$y = (x^2/4 + c)^2$ is a one-parameter family of solutions of the DOE $dy/dx = xy^{1/2}$.

Also $y = 0$ is a solution of this ODE but it is not a member of the family above. It is a singular solution.

The general solution:

If every solution of an nth-order ODE $F(x, y, y', ..., y^{(n)}) = 0$ on an interval I can be obtained from an n-parameter family $G(x, y, c_1, c_2, ..., c_n) = 0$ by appropriate choices of the parameters c_i, $i = 1, 2, ..., n$ we then say that the family is the **general solution** of the differential equation.

Systems of differential equation:

A **system of ordinary differential equations** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable.

Example:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= f(t,x,y) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= g(t,x,y) \end{aligned}$$

A **solution** of a system, such as above, is a pair of differentiable functions $x = \phi_1(t)$ and $y = \phi_2(t)$ defined on a common interval I that satisfy each equation of the system on this interval.

Initial value problem:

On some interval I containing x_0, the problem of solving

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = f\left(x, y, y', ..., y^{(n)}\right)$$

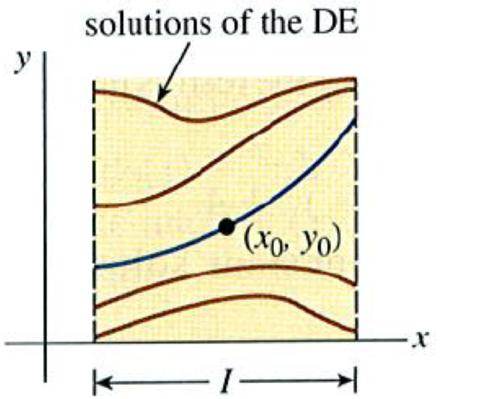
subject to the conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad ..., \quad y^{(n-1)}(x_0) = y_{n-1}$$

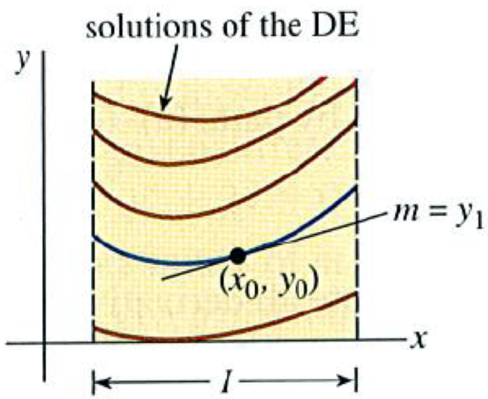
where y_0, y_1, ... , y_{n-1} are arbitrarily specified constants, is called an **initial value problem (IVP)**.

The conditions $y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(n-1)}(x_0) = y_{n-1}$ are called **initial conditions**.

First-order and Second-order IVPs:



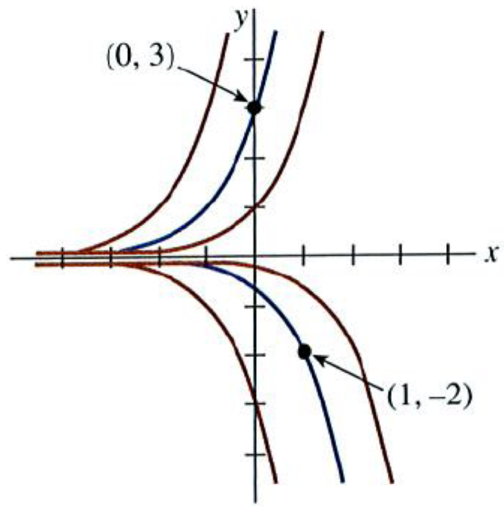
$$\begin{aligned} \frac{dy}{dx} &= f(x,y) \\ y(x_0) &= y_0 \end{aligned} \qquad (3)$$



$$\begin{aligned} \frac{d^2y}{dx^2} &= f(x,y,y') \\ y(x_0) &= y_0 \\ y'(x_0) &= y_1 \end{aligned} \qquad (4)$$

Example:

$y = ce^x$ is a one-parameter family of solutions of the first order ODE $y' = y$ on the interval $(-\infty, \infty)$.



The initial condition $y(0) = 3$ determines the constant c:

$$y(0) = 3 = ce^0 = c$$

Thus the function $y = 3e^x$ is a solution of the IVP defined by

$$y' = y, \quad y(0) = 3$$

Similarly, the initial condition $y(1) = -2$ will yield $c = -2e^{-1}$. The function $y = -2e^{x-1}$ is a solution of the IVP

$$y' = y, \quad y(1) = -2$$

Existence and uniqueness:

Does a solution of the problem exist? If a solution exist, is it unique?

Existence (for the IVP (3)):
Does the differential equation $dy/dx = f(x, y)$ *possess solutions?*
Do any of the solution curves pass through the point (x_0, y_0)*?*

Uniqueness (for the IVP (3)):
When can we be certain that there is precisely one solution curve passing through the point (x_0, y_0)*?*

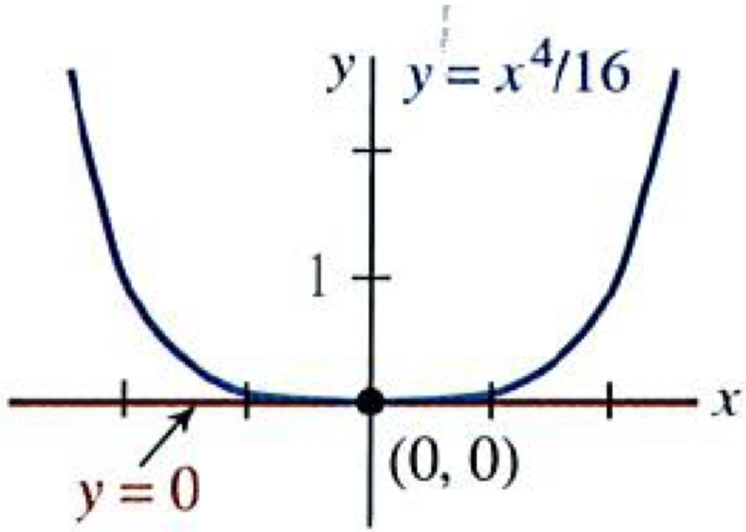
Example: An IVP can have several solutions

Each of the functions

$$\begin{aligned} y &= 0 \\ y &= x^4/16 \end{aligned}$$

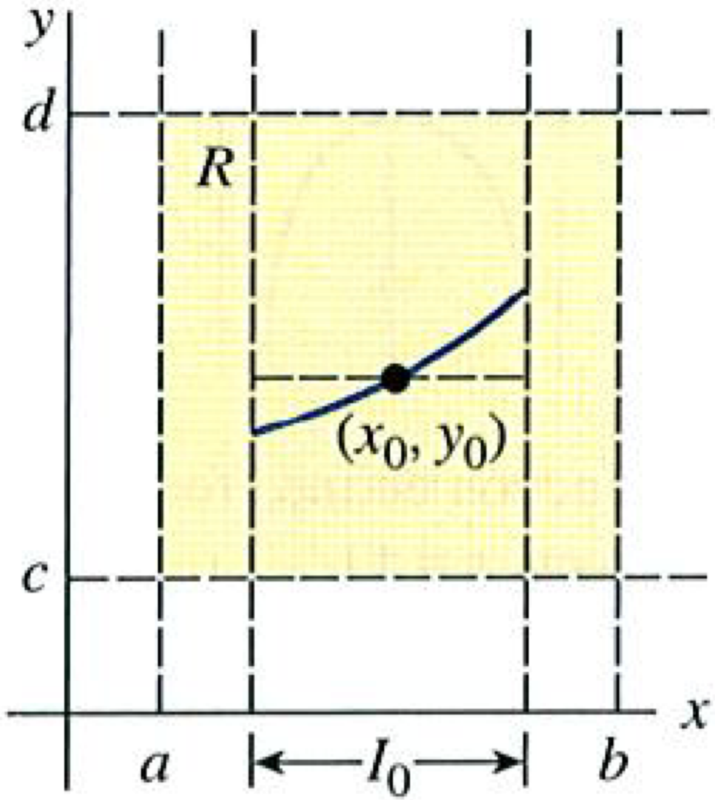
satisfy the IVP

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= xy^{1/2} \\ y(0) &= 0 \end{aligned}$$



Theorem: Existence of a unique solution

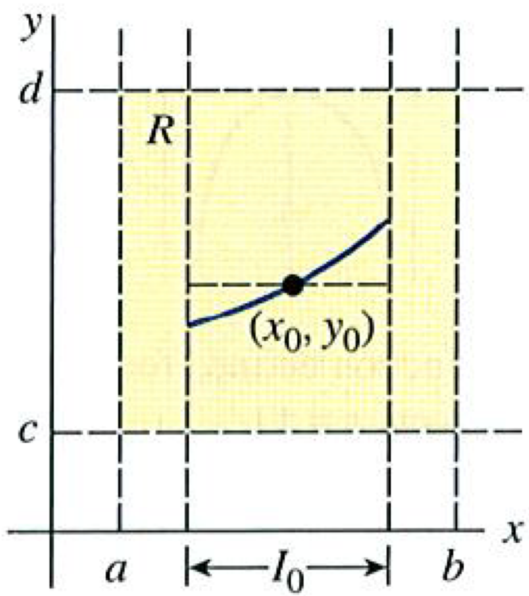
Let R be a rectangular region in the xy-plane defined by $a \le x \le b$, $c \le y \le d$, that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\partial f/\partial y$ are continuous on R, then there exist some interval I_0: $x_0 - h < x < x_0 + h$, $h > 0$, contained in $a \le x \le b$, and a unique function $y(x)$ defined on I_0, that is a solution of the initial value problem (3).



Distinguish the following three sets on the real x-axis:

the domain of the function $y(x)$;
the interval I over which the solution $y(x)$ is defined or exists;
the interval I_0 of existence AND uniqueness.

The theorem above gives no indication of the sizes of the intervals I and I_0; the number $h > 0$ that defines I_0 could be very small. Thus we should think that the solution $y(x)$ is *unique in a local sense*, that is near the point (x_0, y_0).



Example: uniqueness

Consider again the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy^{1/2}$$

in the light of the theorem above. The functions

$$f(x,y) = xy^{1/2}$$
$$\frac{\partial f}{\partial y} = \frac{x}{2y^{1/2}}$$

are continuous in the upper half-plane defined by $y > 0$.

The theorem allow us to conclude that through any point (x_0, y_0), $y_0 > 0$, in the upper half-plane, there is an interval centered at x_0, on which the ODE has a unique solution.

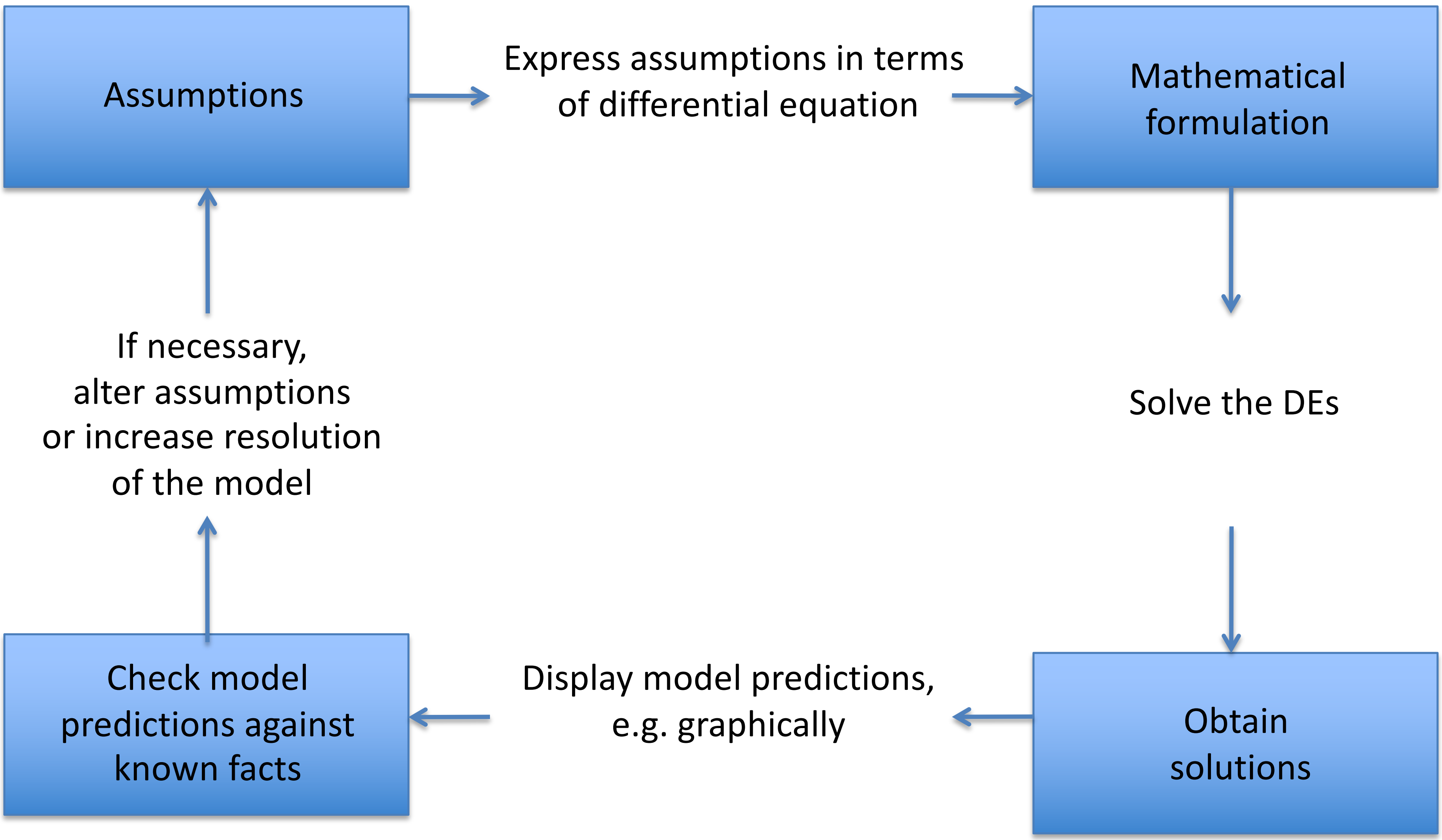
Frank and Ernest

Mathematical model

is the mathematical descriptions of a system or a phenomenon. Construction:

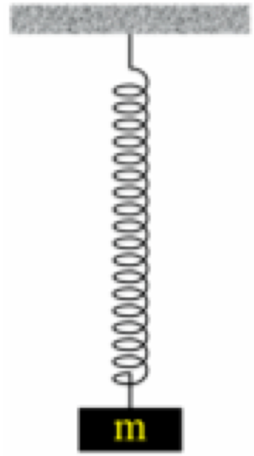
- identifying variables, including specifying the **level of resolution**;
- making a set of reasonable assumptions or hypotheses about the system, including empirical laws that are applicable; these often involve a rate of change of one or more variables and thus differential equation.
- trying to solve the model, and if possible, verifying, improving: increasing resolution, making alternative assumptions etc.

A mathematical model of a physical system will often involve time. A solution of the model then gives the **state of the system**, the values of the dependent variable(s), at a time t, allowing us to describe the system in the past, present and future.


Assumptions
Express assumptions in terms of differential equation
Mathematical formulation
Solve the DEs
Obtain solutions
Display model predictions, e.g. graphically
Check model predictions against known facts
If necessary, alter assumptions or increase resolution of the model

Examples of ordinary differential equations

(1) **Spring-mass problem**



Newton's law

$$F = ma = m\frac{\mathrm{d}v}{\mathrm{d}t} = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$$

Hook's law

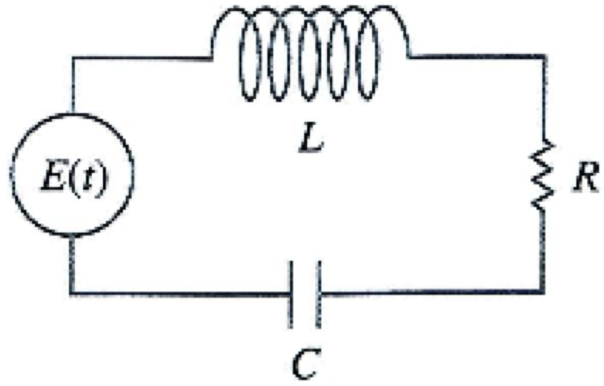
$$F = -kx$$

By putting these two laws together we get the desired ODE

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x \quad = \quad 0$$

where we introduced the angular frequency of oscillation $\omega = \sqrt{k/m}$.

(2) **RLC circuit**



$i(t)$ - the current in a circuit at time t

$q(t)$ - the charge on the capacitor at time t

L - inductance

C - capacitance

R - resistance

According to **Kirchhoff's second law**, the impressed voltage $E(t)$ must equal to the sum of the voltage drops in the loop.

$$V_L + V_C + V_R = E(t)$$

Inductor

$$V_L = L\frac{\mathrm{d}i}{\mathrm{d}t} = L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2}$$

Capacitor

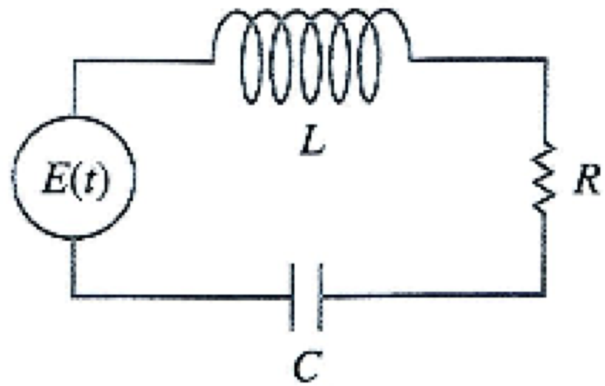
$$V_C = \frac{q}{C}$$

Resistor

$$V_R = Ri = R\frac{\mathrm{d}q}{\mathrm{d}t}$$

RLC circuit

$$L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q = E(t)$$



First-order differential equations

To find either explicit or implicit solution, we need to

(i) recognize the *kind* of differential equation, and then

(ii) apply to it an equation-specific method of solution.

Separable variables

Solution by integration

The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x) \tag{2}$$

is the simplest ODE. It can be solved by integration:

$$y(x) = \int g(x)dx = G(x) + c$$

where $G(x)$ is an indefinite integral of $g(x)$.

Example:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + e^{2x}$$

has the solution

$$y = \int \left(1 + e^{2x}\right) dx = \frac{1}{2}e^{2x} + x + c.$$

This ODE and its method of solution is a special case when f is a product of a function of x and a function of y.

Definition: Separable equation

A first-order differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y) \tag{3}$$

is said to be **separable** or to have **separable variables**.

Method of solution:

A one parameter family of solutions, usually given implicitly, is obtained by first rewriting the equation in the form

$$p(y)dy = g(x)dx$$

where $p(y) = 1/h(y)$, and integrating both sides of the equation. We get the solution in the form

$$H(y) = G(x) + c$$

where $H(y) = \int p(y)dy$ and $G(x) = \int g(x)dx$ and c is the combined constant of integration.

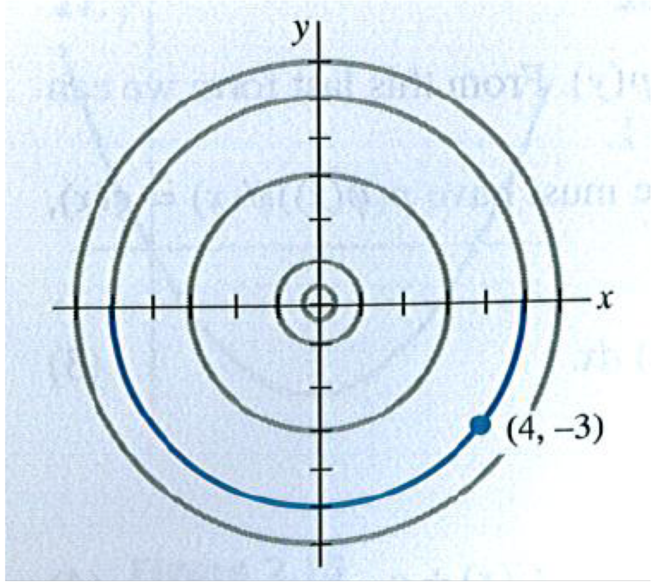
Example: A separable ODE

Solve

$$(1+x)dy - ydx = 0$$

Dividing by $(1+x)y$ we get $dy/y = dx/(1+x)$ and can integrate

$$\begin{aligned}
\int \frac{dy}{y} &= \int \frac{dx}{1+x} \\
\ln|y| &= \ln|1+x| + c_1 \\
y &= e^{\ln|1+x|+c_1} = e^{\ln|1+x|}.e^{c_1} \\
&= |1+x|\, e^{c_1} \\
&= \pm e^{c_1}(1+x) = c(1+x)
\end{aligned}$$



Example: Solution curve

Solve the initial value problem

$$\frac{dy}{dx} = -\frac{x}{y}, \qquad y(4) = -3$$

By rewriting the equation as $ydy = -xdx$, we get

$$\begin{aligned} \int ydy &= -\int xdx \\ \frac{y^2}{2} &= -\frac{x^2}{2} + c_1 \end{aligned}$$

We can rewrite the result as $x^2 + y^2 = c^2$, where $c^2 = 2c_1$. This family of solutions represents a family of concerning circles centered at the origin. The IVP determines the circle $x^2 + y^2 = 25$ with radius 5.

Losing a solution

Some care should be exercised when separating variables, since the variable divisors could be zero at a point.

If r is a zero of $h(y)$, then substituting $y = r$ into $dy/dx = g(x)h(y)$ makes both sides zero, i.e. $y = r$ is a constant solution of the DE.

This solution, which is a singular solution, can be missed in the course of the solving the ODE.

Example:

Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 4$$

We put the equation into the following form by using partial fractions

$$\frac{dy}{y^2 - 4} = \left[\frac{1/4}{y-2} - \frac{1/4}{y+2}\right] dy = dx$$

and integrate

$$\frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2| = x + c_1$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4x + c_2$$

$$\frac{y-2}{y+2} = e^{4x+c_2}$$

We substitute $c = e^{c_2}$ and get the one-parameter family of solutions

$$y = 2\frac{1 + ce^{4x}}{1 - ce^{4x}}$$

Actually, if we factor the r.h.s. of the ODE as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (y - 2)(y + 2)$$

we see that $y = 2$ and $y = -2$ are two constant (equilibrium solutions). The earlier is a member of the family of solutions defined above corresponding to $c = 0$. However $y = -2$ is a singular solution and in this example it was lost in the course of the solution process.

Example: an IVP

Solve

$$\cos x\left(e^{2y}-y\right)\frac{\mathrm{d}y}{\mathrm{d}x} \;=\; e^{y}\sin 2x, \qquad y(0)=0$$

By dividing the equation we get

$$\frac{e^{2y}-y}{e^{y}}dy \;=\; \frac{\sin 2x}{\cos x}dx$$

We use the trigonometric identity $\sin 2x = 2\sin x\cos x$ on r.h.s. and integrate

$$\begin{aligned}\int\left(e^{y}-ye^{-y}\right)dy &= 2\int \sin x dx \\ e^{y}+ye^{-y}+e^{-y} &= -2\cos x + c\end{aligned}$$

The initial condition $y(0)=0$ implies $c=4$, so we get the solution of the IVP

$$e^{y}+ye^{-y}+e^{-y} \;=\; 4-2\cos x$$

Example: an IVP

Solve

$$\cos x\left(e^{2y}-y\right)\frac{\mathrm{d}y}{\mathrm{d}x} = e^{y}\sin 2x, \qquad y(0)=0$$

By dividing the equation we get

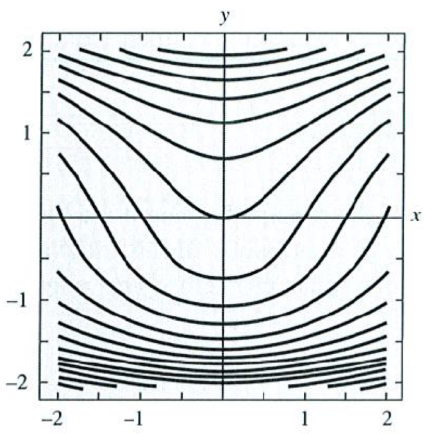
$$\frac{e^{2y}-y}{e^{y}}dy = \frac{\sin 2x}{\cos x}dx$$

We use the trigonometric identity $\sin 2x = 2\sin x\cos x$ on r.h.s. and integrate

$$\int\left(e^{y}-ye^{-y}\right)dy = 2\int \sin x dx$$

$$e^{y}+ye^{-y}+e^{-y} = -2\cos x + c$$

The initial condition $y(0)=0$ implies $c=4$, so we get the solution of the IVP



$$e^{y}+ye^{-y}+e^{-y} = 4-2\cos x$$